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# ELASTIC PLATE REINFORCED BY SYSTEM OF SLENDER CORDS

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Summary

Using the results of [1] the governing system of relations for an elastic plate reinforced by a system of slender cords has been derived. The solution to the axially-symmetric boundary value plane problem has been obtained and discussed.

1. Introduction. Let $\Omega$ be the regular region in $R^2$ occupied in the reference configuration by the elastic plate, which is reinforced by system of cords. We assume that the cords coinside with curves $\Gamma_k$, $k = 1, \ldots, n$. We denote by $t_k(x)$, $n_k(x)$, $x \in \Gamma_k$ the fields of unit vectors tangent and normal to the curve $\Gamma_k$, respectively. Let $s_k(x)$ and $e_k(x)$ be the values of tension and strain, respectively in cords $\Gamma_k$. We assume that the cords are slender, so the values of tension in cords are restricted by the conditions $s_k \geqslant 0$.

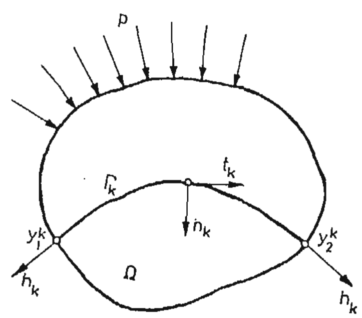


Fig. 1.

We are to derive, within the range of the linear static elasticity, the local relations describing boundary value problem for a plate reinforced by a system of slender cords. Using the obtained local relations we are to solve axially-symmetric boundary value problem for an elastic circular plate reinforced by a system of slender cords.

The statring point of the analysis is the system of relations which have been proposed in [1]; namely:

1° global equilibrium equation

$$\int_{\Omega} \operatorname{tr}[S(x)Lv(x)]ds + \sum_{k=1}^{n} \int_{\Gamma_k} \operatorname{tr}[t_k(x)\otimes t_k(x)Lv(x)]s_k(x)dl =$$

$$= \int_{\partial\Omega} p(x)v(x)dl + \int_{\Omega} b(x)v(x)ds, \qquad \forall v \in V, \tag{1.1}$$

2° constitutive relations

$$\begin{aligned} E(x) &= \mathscr{K}(x)S(x) \\ e_k(x) - K_k(x)s_k(x) &\in \partial\chi_{\bar{R}^+}(S_k(x)) \end{aligned} \tag{1.2}$$

3° strain — displacement relations

$$\begin{aligned} E(x) &= Lu(x), \qquad Lv \equiv \frac{1}{2}(\nabla v + (\nabla v)^T), \\ e_k(x) &= \operatorname{tr}[t_k(x)\otimes t_k(x)Lu(x)], \end{aligned} \tag{1.3}$$

where: $S(x)$ is the stress tensor, $b(x)$ is the body force, $p(x)$ is the surface force, $\chi_{\bar{R}^+}(s)$ is the indicator function of $\bar{R}^+$ e. a.

$$\chi_{R^+}(s) = \begin{cases} 0 & \text{for} \quad s \in R^+ \\ +\infty & \text{for} \quad s \notin R^+ \end{cases}$$

and $\delta\chi_{R^+}(s)$ is a subdifferential on indicator function at the point $s$.

2. Basic relations. In order to obtain the local relations from the global equilibroum equation (1.1) we transform the integrals appearing in Eq. (1.1)

$$\int_{\Omega} \operatorname{tr}[S(x)Lv(x)]ds = \int_{\partial\Omega} S^{ij}v_i n_j dl - \int_{\Omega} S^{ij}_{,j} v_i dx^1 dx^2 + \sum_{k=1}^{n} \int_{\Gamma_k} [\![S^{ij}]\!] v_i n_{kj} dl \tag{2.1}$$

and

$$\int_{\Gamma_k} \operatorname{tr}[t_k\otimes t_k Lv]s_k dl = s_k t^i v_i|_{x=y_2^k} - s_k t_k^i v_i|_{x=y_1^k} - \int_{\Gamma_k} \left(\varkappa_k n_k^i s_k + t_k^i \frac{ds_k}{dl}\right) v_i dl, \tag{2.2}$$

where $[\![S^{ij}]\!] \equiv S^{ij}_{+} - S^{ij}_{-}$, $\quad S^{ij}_{+}(y) = \lim\limits_{\substack{x\to y \\ (x-y)n_k>0}} S^{ij}(x)$, $\quad S^{ij}_{-}(y) = \lim\limits_{\substack{x\to y \\ (x-y)n_k<0}} S^{ij}(x)$,

$\varkappa_k$ is the curvature of curve $\Gamma_k$, symbol $d(\cdot)/dl$ denotes derivative in the direction tangent to a curve, $y_1^k$, $y_2^k$ are the points of intersection curve $\Gamma_k$ and boundary $\partial\Omega$ of the region $\Omega$.

The global equilibrium equation have to be satisfied for arbitrary functions $v \in V$; Hence using (2.1) and (2.2), we arrive at local relations

$$\begin{aligned} S^{ij}_{,j} - b^i &= 0, & x &\in \Omega \setminus \bigcup_k \Gamma_k, \\ S^{ij}n_j &= -p^i, & x &\in \partial\Omega, \\ \varkappa_k s_k n_k^i + t_k^i \frac{ds_k}{dl} &= -[\![S^{ij}]\!]n_{kj}, & x &\in \Gamma_k, \\ s_k &= -h_k, & x &\in \Gamma_k \cap \partial\Omega, \end{aligned} \tag{2.3}$$

where $h_k$ is the known value of a traction in the cord at the boundary (in the tangent direction $t_k$).

Eqs. (1.2), (1.3) can be written as

$$\begin{aligned} E_{ij} &= \mathscr{K}_{ijlm} S^{lm}, \\ e_k &= K_k s_k, \quad \text{if} \quad s_k > 0, \\ e_k &\leqslant 0, \quad \text{if} \quad s_k = 0, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} E_{ij} &= \frac{1}{2}(u_{i,j}+u_{j,i}), \\ e_k &= t_k^i t_k^j u_{i,j} \end{aligned} \tag{2.5}$$

respectively.

Eqs. (2.3), (2.4), (2.5) constitute the basic system of relations descvibing boundary — value plane problem for a plate reinforced by a system of slender cords.

3. Example. Let the elastic circular plate with the concentric hole be isotropic and homogeneous. We assume that the plate is reinforced by a system of slender cords which coincide with curves $\varrho = \zeta_k = \text{const.}$ $k = 1, \ldots, n-1$, $\zeta_k \in (\text{a, b})$. Let $\zeta_0 = a$ and $\zeta_n = b$, and let the plate be loaded at the boundary $\varrho = a$ and $\varrho = b$ by the known radial tractions $p_a$ and $p_b$, respectively. In order to obtain displacements, stresses, strains and tractions in cords we shall use the relations obtained in Sec. 2. Takeing into acount the axial symmetry of the pertinent problem we shall writte relations (2.3), (2.4) and (2.5) in the polar coordinate system as follows:

1° equilibrium equations

$$\begin{aligned} &(\varrho S_\varrho(\varrho))_{,\varrho} - S_\Theta(\varrho) = 0 \\ &S_\varrho(a) = -p_a, \quad S_\varrho(b) = -p_b \\ &\frac{1}{\zeta_k} s_k = -[\![ S_\varrho(\zeta_k) ]\!], \quad s_k \geqslant 0, \end{aligned} \tag{3.1}$$

where $S_\varrho$ and $S_\Theta$ are the main values of the stress, tensor,

2° constitutive relations

$$\begin{aligned} E_\varrho &= \frac{1}{E}(S_\varrho - \nu S_\Theta), \\ E_\Theta &= \frac{1}{E}(S_\Theta - \nu S_\varrho), \\ e_k &= \begin{cases} K s_k & \text{for} \quad s_k > 0 \\ \delta_k \leqslant 0 & \text{for} \quad s_k = 0. \end{cases} \end{aligned} \tag{3.2}$$

3° strain — displacement relations

$$E_\varrho = w_{,\varrho}, \quad E_\Theta = \frac{w}{\varrho}, \quad e_k = \frac{w(\zeta_k)}{\zeta_k}. \tag{3.3}$$

In order to obtain solution of the problem given by Eqs. (3.1), (3.2), (3.3), we consider separately the following four cases:

(i) All cords are unstressed:

$$s_k = 0, \quad e_k \leqslant 0, \quad \text{for} \quad k = 1, \ldots, n-1,$$

(ii) All cords are stressed:

$$s_k > 0, e_k = Ks_k, \quad \text{for} \quad k = 1, \ldots, n-1,$$

(iii) Cords $\zeta_1, \ldots, \zeta_{\bar{k}-1}$ are unstressed and cords $\zeta_{\bar{k}}, \ldots, \zeta_{n-1}$ are stressed:

$$s_k = 0, \quad e_k \leqslant 0, \quad \text{for} \quad k = 1, \ldots, \bar{k}-1,$$

$$s_k > 0, \quad e_k = Ks_k, \quad \text{for} \quad k = \bar{k}, \ldots, n-1,$$

(iv) Cords $\zeta_1, \ldots, \zeta_{\bar{k}-1}$ are stressed but cords $\zeta_{\bar{k}}, \ldots, \zeta_{n-1}$ are not:

$$s_k > 0, \quad e_k = Ks_k, \quad \text{for } k = 1, \ldots, \bar{k}-1,$$

$$s_k = 0, \quad e_k \leqslant 0, \quad \text{for } k = \bar{k}, \ldots, n-1.$$

(i) Assume that $s_k = 0$ hold for $k = 1, \ldots, n-1$. From $(3.1)_3$ we see, that $S_\varrho$ is a continuous function at points $\zeta_k$. Eqs. $(3.1)_{1,2}$, $(3.2)_{1,2}$ and $(3.3)_{1,2}$ represent the system of equations for an elastic plate. In this case the following solution holds

$$S_\varrho = A_1 \frac{1}{\varrho^2} + 2C_1, \qquad S_\Theta = -A_1 \frac{1}{\varrho^2} + 2C_1,$$
$$E_\Theta = \frac{1-\nu}{E}\left(2C_1 - A_1 \frac{1+\nu}{1-\nu}\frac{1}{\varrho^2}\right), \qquad w = E_\Theta \varrho. \tag{3.4}$$

The constants $A_1$, $C_1$ can be found from boundary conditions $(3.1)_2$

$$A_1 = \frac{a^2 b^2 (p_b - p_a)}{b^2 - a^2}, \qquad 2C_1 = \frac{p_a a^2 - p_b b^2}{b^2 - a^2}. \tag{3.5}$$

Relations (3.4) represent the solution to the problem under consideration provided that strains $e_k$ in cords $\Gamma_k$, which are due to loadings $p_a$ and $p_b$ satisfy conditions

$$e_k = E_\Theta(\zeta_k) = \frac{w(\zeta_k)}{\zeta_k} \leqslant 0.$$

From aforementional conditions we get

$$p_b \geqslant \frac{\lambda_k^b}{\lambda_k^a} p_a, \tag{3.6}$$

where

$$\lambda_k^a = \frac{1}{a^2} + \frac{\mathring{\nu}}{\zeta_k^2}, \qquad \lambda_k^b = \frac{1}{b^2} + \frac{\mathring{\nu}}{\zeta_k^2}, \qquad \mathring{\nu} = \frac{1+\nu}{1-\nu}.$$

Since $\lambda_{k+1}^b/\lambda_{k+1}^a \leqslant \lambda_k^b/\lambda_k^a$ for $k = 1, \ldots, n-2$, then it follows that the inequalities (3.6) are satysfied if

$$(p_a, p_b) \in B_1 = B_1^0 \cup B_1^n, \tag{3.7}$$

where

$$B_1^0 = \left\{(p_a, p_b) \in R^2; p_a > 0, p_b \geqslant p_a \frac{\lambda_1^b}{\lambda_1^a}\right\},$$

$$B_1^n = \left\{(p_a, p_b) \in R^2; \quad p_a \leqslant 0, \quad p_b \geqslant p_a \frac{\lambda_{n-1}^b}{\lambda_{n-1}^a}\right\}.$$

If radial tractions $p_a$, $p_b$ satisfy the condition (3.7) then the solution to the pertinent problem is given by the relation (3.4) in which constants $A_1$, $C_1$ are determined by Eqs. (3.5).

(ii) If $s_k > 0$ holds for $k = 1, \dots, n-1$, then form $(3.1)_3$ it follows that the stress $S_\varrho$ is discontiuons for $\varrho = \zeta_k$, $k = 1, \dots, n-1$. Then the system of Eqs. $(3.1)_1$, $(3.2)_{1,2}$, $(3.3)_{1,2}$ has the solution in every interval $(\zeta_{k-1}, \zeta_k)$ in the form

$$S_\varrho = \frac{A_k}{\varrho^2}+2C_k, \qquad S_\Theta = -\frac{A_k}{\varrho^2}+2C_k,$$
$$E_\Theta = \frac{1-\nu}{E}\left(2C_k-\frac{\mathring{\nu}}{\varrho^2}A_k\right), \qquad w = \varrho E_\Theta. \tag{3.8}$$

From Eqs. $(3.1)_3$, $(3.8)_1$ we get

$$s_k = \zeta_k(2C_{k+1}-2C_k)+\frac{1}{\zeta_k}(A_{k+1}-A_k) \tag{3.9}$$

and from Eqs. $(3.3)_3$, $(3.8)_{1,2}$ it follows that

$$e_k = \frac{1-\nu}{E}\left(2C_k-\frac{\mathring{\nu}}{\zeta_k^2}A_k\right). \tag{3.10}$$

In order to determine constants $A_k$, $C_k$ we take into account boundary conditions $(3.1)_2$, equations $e_k = Ks_k$, $k = 1, \dots, n-1$ and the continuity conditions for the displacement field. The continuity conditions for displacement at every $\varrho = \zeta_k$ and equations $e_k = Ks_k$, $k = 1, \dots, n-1$, leads to the following recurrent relations for constants $A_k$, $C_k$

$$A_{k+1} = \gamma_k A_k+\delta_k 2C_k, \; 2C_{k+1} = \alpha_k A_k+\beta_k 2C_k \tag{3.11}$$

where:

$$\alpha_k = -\frac{\mathring{\nu}^2}{\zeta_k^3}\bar{K}, \qquad \beta_k = 1+\frac{\mathring{\nu}}{\zeta_k}\bar{K},$$
$$\gamma_k = 1-\frac{\mathring{\nu}}{\zeta_k}\bar{K}, \qquad \delta_k = \zeta_k\bar{K}, \qquad \bar{K} = \frac{(1-\nu)^2}{2EK}. \tag{3.12}$$

From (3.11) constants $A_k$, $C_k$, $k = 2, \dots, n$ can be expressed in term of constants $A_1$, $C_1$

$$A_k = c_k A_1+d_k 2C_1, \qquad 2C_k = a_k A_1+b_k 2C_1, \tag{3.13}$$

where

$$a_{k+1} = \alpha_k c_k+\beta_k a_k, \qquad b_{k+1} = \alpha_k d_k+\beta_k b_k,$$
$$c_{k+1} = \gamma_k c_k+\delta_k a_k, \qquad d_{k+1} = \gamma_k d_k+\delta_k b_k, \tag{3.14}$$
$$a_1 = 0, b_1 = 1, \qquad c_1 = 1, d_1 = 0.$$

From boundary conditions $(3.1)_2$ we obtain

$$\frac{1}{a^2}A_1+2C_1 = -p_a, \qquad \frac{1}{b^2}A_n+2C_n = p_b. \tag{3.15}$$

Taking into account (3.13), we get

$$\frac{1}{a^2}A_1+2C_1 = -p_a, \qquad \nu_n A_1+\mu_n 2C_1 = -p_0 \tag{3.16}$$

where $\nu_n = 1/b^2\; c_n+a_n$, $\mu_n = 1/b^2\; d_n+b_n$.

The forementioned system of equations has the solution of the form

$$A_1 = \frac{1}{\Delta}(p_b - \mu_n p_a), \quad 2C_1 = \frac{1}{\Delta}\left(\nu_n p_a - \frac{1}{a^2} p_b\right) \tag{3.17}$$

where $\Delta = 1/a^2\mu_n - \nu_n$.
Constants $A_k$, $C_k$ will be given by

$$\begin{aligned} A_k &= \frac{1}{\Delta}\left[p_b\left(c_k - \frac{1}{a^2} d_k\right) + p_a(d_k\nu_n - \mu_n c_k)\right], \\ 2C_k &= \frac{1}{\Delta}\left[p_b\left(a_k - \frac{1}{a^2} b_k\right) + p_a(b_k\nu_n - \mu_n a_k)\right]. \end{aligned} \tag{3.18}$$

Now, the range of the boundary tractions $p_a$, $p_b$ should be determined such that inequalities $s_k > 0$, $k = 1, \ldots, n-1$ hold. We can prove that sequences of constants $a_k$, $b_k$, $c_k$, $d_k$, which are given by (3.14) are monotonic

$$\begin{aligned} a_{k+1} < a_k < 0, \quad 1 < b_k < b_{k+1}, \\ c_{k+1} < c_k < 1, \quad 0 < d_k < d_{k+1} \end{aligned} \tag{3.19}$$

Hence

$$\nu_n < \frac{1}{b^2}, \quad \mu_n > 1, \quad \Delta > 0. \tag{3.20}$$

Let us introduce the following denotations

$$\begin{aligned} r_k &= a_k - \mathring{\nu}\frac{1}{\zeta_k^2} c_k, \quad & q_k &= b_k - \mathring{\nu}\frac{1}{\zeta_k^2} d_k, \\ l_k^a &= \frac{1}{a^2} q_k - r_k, \quad & l_k^b &= q_k\nu_n - r_k\mu_n, \end{aligned} \tag{3.21}$$

and note that $l_k^a > 0$. Conditions $s_k > 0$ lead to the range of boundary tractions as follows

$$p_b < p_a \frac{l_k^b}{l_k^a}, \quad k = 1, \ldots, n-1, \tag{3.22}$$

because $l_{k+1}^b/l_{k+1}^a < l_k^b/l_k^a$ and $l_1^a = \lambda_1^a$, inequality (3.22) fulfield for each k $= 1, \ldots, n-1$ if

$$(p_a, p_b) \in B_2 = B_2^0 \cup B_2^n, \tag{3.23}$$

where

$$\begin{aligned} B_2^0 &= \left\{(p_a, p_b) \in R^2; \quad p_a < 0, \quad p_b < p_a \frac{l_1^b}{l_1^a}\right\}, \\ B_2^n &= \left\{(p_a, p_b) \in R^2; \quad p_a > 0 \quad p_b < p_a \frac{l_{n-1}^b}{l_{n-1}^a}\right\}. \end{aligned}$$

If tractions $p_a$, $p_b$ satisfy the condition (3.23), then the solution to the pertinent problem is given by relations (3.8), (3.9), (3.10) in which constants $A_k$, $C_k$ are determined by Eqs. (3.18).

(iii) If $s_k = 0$ for $k = 1, \ldots, \bar{k}-1$ and $s_k > 0$ for $k = \bar{k}, \ldots, n-1$ then the stress $S_\varrho$ is continuous on the interval $(a, \zeta_{\bar{k}})$ and discontinuous for $\varrho = \zeta_k$, $k = \bar{k}, \ldots, n-1$. System of equations $(3.1)_1$, $(3.2)_{1,2}$, $(3.3)_{1,2}$ has the solution given by (3.8) in which for $\varrho \in (a, \zeta_{\bar{k}})$, we have to substitute $k = \bar{k}$ and for $\varrho \in (\zeta_{i-1}, \zeta_i)$, $i = \bar{k}+1, \ldots, n$ we have to substitute $k = i$ the values of tension in cords $\Gamma_k$, $k = \bar{k}, \ldots, n-1$ are determined by relation (3.9) and the values of strain $e_k$ by relation (3.10). Constants $A_k$, $C_k$, $k = \bar{k}, \ldots, n$, which appear in the solution, are determined analogously as befor, i.e., from boundary conditions $(3.1)_2$, Eqs. $e_k = Ks_k$ for $k = \bar{k}, \ldots, n-1$ and from continuity condition for displacement. The continuity condition for the displacement field and equations: $e_k = Ks_k$ lead to recurent relations (3.11) for $k = \bar{k}, \ldots, n-1$. From these relations constants $A_i$, $C_i$, $i = \bar{k}+1, \ldots, n$ can be expressed in term of constants $A_{\bar{k}}$, $C_{\bar{k}}$.

$$\begin{aligned} A_i &= c_i(\bar{k})A_{\bar{k}} + d_i(\bar{k})2C_{\bar{k}}, \\ 2C_i &= a_i(\bar{k})A_{\bar{k}} + b_i(\bar{k})2C_{\bar{k}}, \end{aligned} \tag{3.24}$$

where:

$$\begin{aligned} a_{i+1} &= \alpha_i c_i(\bar{k}) + \beta_i a_i(\bar{k}), \quad & b_{i+1} &= \alpha_i d_i(\bar{k}) + \beta_i b_i(\bar{k}), \\ c_{i+1} &= \gamma_i c_i(\bar{k}) + \delta_i a_i(\bar{k}), \quad & d_{i+k} &= \gamma_i d_i(\bar{k}) + \delta_i b_i(\bar{k}), \\ a_{\bar{k}}(\bar{k}) = 0, \quad b_{\bar{k}}(\bar{k}) &= 1, \quad & c_{\bar{k}}(\bar{k}) = 1, \quad d_{\bar{k}}(\bar{k}) &= 0. \end{aligned} \tag{3.25}$$

From boundary conditions $(3.1)_2$ we get

$$\frac{A_k}{a^2} + 2C_k = -p_a, \qquad \frac{A_n}{b^2} + 2C_n = -p_b. \tag{3.26}$$

Using (3.24) we obtain the system of equation for $A_{\bar{k}}$, $C_{\bar{k}}$, solution of which is given by

$$A_{\bar{k}} = \frac{1}{\bar{\Delta}}(p_b - \mu_n(\bar{k})p_a), \qquad 2C_{\bar{k}} = \frac{1}{\bar{\Delta}}\left(\nu_n(\bar{k})p_a - \frac{1}{a^2}p_b\right), \tag{3.27}$$

where:

$$\nu_n(\bar{k}) = \frac{1}{b^2}c_n(\bar{k}) + a_n(\bar{k}), \qquad \mu_n(\bar{k}) = \frac{1}{b^2}d_n(\bar{k}) + b_n(\bar{k}),$$

$$\bar{\Delta} = \frac{1}{a^2}\mu_n(\bar{k}) - \nu_n(\bar{k}).$$

Constants $A_i$, $C_i$ $i = \bar{k}, \ldots, n$ can be expressed in the form

$$\begin{aligned} A_i &= \frac{1}{\bar{\Delta}}\left[p_b\left(c_i(\bar{k}) - \frac{1}{a^2}d_i(\bar{k})\right) + p_a\left(d_i(\bar{k})\nu_n(\bar{k}) - c_i(\bar{k})\mu_n(\bar{k})\right)\right] \\ 2C_i &= \frac{1}{\bar{\Delta}}\left[p_b\left(a_i(\bar{k}) - \frac{1}{a^2}b_i(\bar{k})\right) + p_a\left(b_i(\bar{k})\nu_n(\bar{k}) - a_i(\bar{k})\mu_n(\bar{k})\right)\right]. \end{aligned} \tag{3.28}$$

From conditions $s_i > 0$, $i = \bar{k}, \ldots, n-1$ we obtain the following restriction on the boundary tractions

$$p_b < p_a \frac{l_i^n(\bar{k})}{l_i^a(\bar{k})}, \qquad i = \bar{k}, \ldots, n-1, \tag{3.29}$$

where:

$$l_i^a(\bar{k}) = \frac{1}{a^2} q_i(\bar{k}) - r_i(\bar{k}), \qquad l_i^n(\bar{k}) = \nu_n(\bar{k}) q_i(\bar{k}) - \mu_n(\bar{k}) r_i(\bar{k}),$$

$$r_i(\bar{k}) = a_i(\bar{k}) - \mathring{\nu} \frac{1}{\zeta_i^2} c_i(\bar{k}), \qquad q_i(\bar{k}) = b_i(\bar{k}) - \mathring{\nu} \frac{1}{\zeta_i^2} d_i(\bar{k}).$$

In the case under discussion, conditions

$$e_i = \frac{1-\nu}{E}\left(2C_{\bar{k}} - \frac{\mathring{\nu}}{\zeta_i^2} A_{\bar{k}}\right) \leqslant 0 \tag{3.30}$$

ought to be satysfied for $i = 1, \ldots, \bar{k}-1$. From these conditions we get inequalities

$$p_b \geqslant p_a \frac{\nu_n(\bar{k}) + \frac{\mathring{\nu}}{\zeta_i^2} \mu_n(\bar{k})}{\lambda_i^a} \qquad i = 1, \ldots, \bar{k}-1 \tag{3.31}$$

Conditions (3.29) and (3.31) are fulfield if

$(p_a, p_b) \in B_3^k$,

$$B_3^k = \{(p_a, p_b) \in R^2; \quad p_a \leqslant 0, \quad p_a l_{k-1}^n / \lambda_{k-1}^a \leqslant p_b < p_a l_k^n / \lambda_k^a\}. \tag{3.32}$$

If boundary tractions $p_a$, $p_b$ satisfy the condition

$$(p_a, p_b) \in B_3 = \bigcup_{k=1}^{n-1} B_3^k \tag{3.33}$$

then the solution to the problem under consideration is given by relations (3.8), (3.9), (3.10) in which constants $A_k$, $C_k$, $k = 1, \ldots, n$ are determined by (3.28) and wehere $A_k = A_{\bar{k}}$, $C_k = C_{\bar{k}}$ for $k = 1, \ldots, \bar{k}-1$.

(iv) In this case $s_k > 0$ for $k = 1, \ldots, \bar{k}-1$ and $s_k = 0$ for $k = \bar{k}, \ldots, n-1$. Then the stress $S_\varrho$ is discontinuous for $\varrho = \zeta_i$, $i = 1, \ldots, k-1$. System of Eqs. $(3.1)_1$, $(3.2)_{1,2}$, $(3.3)_{1,2}$ have the solution given by (3.8), in which for $\varrho \in (\zeta_{i-1}, \zeta_i)$, $i = 1, \ldots, \bar{k}-1$ we have to substitute $k = i$ and for $\varrho \in (\zeta_{\bar{k}-1}, b)$ we have to substitute $k = \bar{k}$. Constants $A_k$, $C_k$, $k = 1, \ldots, \bar{k}$ which appear in the solution are determined by the procedure analogous to that given above. From the ciontinuity condition for the displacement field and from relations $e_k = Ks_k$, $k = 1, \ldots, \bar{k}-1$, we obtain reccurrent relations (3.11) for $k = 1, \ldots, \bar{k}-1$. and hence (3.14) holds.

From boundary conditica $(3.1)_2$ we get

$$\frac{A_1}{a^2} + 2C_1 = -p_a, \qquad \frac{1}{b^2} A_{\bar{k}} + 2C_{\bar{k}} = -p_b \tag{3.34}$$

Substituting (3.14) into (3.34), we obtain the system of equations for $A_1$ and $C_1$, solution of which is

$$A_1 = \frac{1}{\Delta_{\bar{k}}}(p_b - \mu_{\bar{k}} p_a), \qquad 2C_1 = \frac{1}{\Delta_{\bar{k}}}\left(\nu_{\bar{k}} p_a - \frac{1}{a^2} p_b\right)$$

where $\quad \Delta_{\bar{k}} = 1/a^2 \mu_{\bar{k}} - \nu_{\bar{k}}$.

The formulas for constants $A_k$, $C_k$ have the form

$$
\begin{aligned}
A_k &= \frac{1}{\Delta_{\bar{k}}}\left[p_b\left(c_k-\frac{1}{a^2}\,d_k\right)+p_a(d_k\nu_{\bar{k}}-c_k\mu_{\bar{k}})\right]\\
2C_k &= \frac{1}{\Delta_{\bar{k}}}\left[p_b\left(a_k-\frac{1}{a^2}\,b_k\right)+p_a(b_k\nu_{\bar{k}}-a_k\mu_{\bar{k}})\right]
\end{aligned}
\tag{3.35}
$$

From inequalities $s_k > 0$, $k = 1, \ldots, \bar{k}-1$ we obtain the following restriction for loadings

$$
p_b < p_a\frac{l_i^{\bar{k}}}{l_i^a} \qquad \text{for} \qquad i = 1, \ldots, \bar{k}-1 \tag{3.36}
$$

where

$$
l_i^{\bar{k}} = q_i\nu_{\bar{k}}-r_i\mu_{\bar{k}}, \qquad l_i^a = \frac{1}{a^2}\,q_i-r_i.
$$

From conditions

$$
e_i = \frac{1-\nu}{E}\left(2C_{\bar{k}}-\frac{\mathring{\nu}}{\zeta_i^2}A_{\bar{k}}\right) \leqslant 0, \qquad i = \bar{k}, \ldots, n-1
$$

we obtain

$$
p_b \geqslant p_a\frac{\lambda_{\bar{k}}^b}{l_{\bar{k}}^a} \tag{3.37}
$$

Conditions (3.36) and (3.37) are fulfield if

$$
(p_a, p_b) \in B_4^{\bar{k}}, \qquad B_4^{\bar{k}} = \left\{(p_a, p_b) \in R^2; \quad p_a > 0, \quad p_a\frac{\lambda_{\bar{k}}^b}{l_{\bar{k}}^a} \leqslant p_b < p_a\frac{\lambda_{\bar{k}-1}^b}{l_{\bar{k}-1}^a}\right\}. \tag{3.38}
$$

If boundary tractions $p_a$, $p_b$ satisfy the condition

$$
(p_a, p_b) \in B_4 = \bigcup_{k=1}^{n-1} B_4^k \tag{3.39}
$$

then the solution is given by relations (3.8), (3.9), (3.10) in which constants $A_k$, $C_k$, $k = 1, \ldots, \bar{k}$ are determine by (3.35), and where $A_k = A_{\bar{k}}$, $C_k = C_{\bar{k}}$ for $k = \bar{k}+1, \ldots, n$.

Note that every pair from sets $B_1$, $B_2$, $B_3$, $B_4$ is disjointed and that $B_1 \cup B_2 \cup B_3 \cup B_4 = R^2$. For the case $n = 4$ the aforementioned situation is ilustrated on Fig. 2.

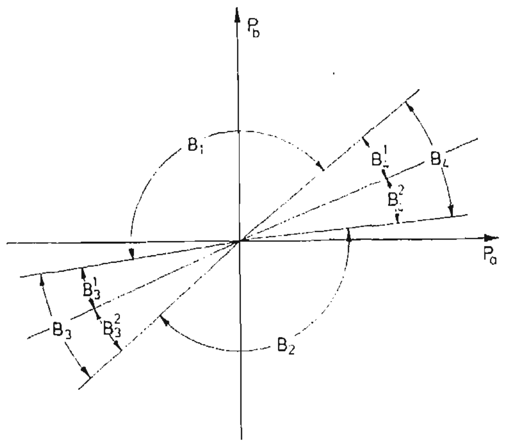


Fig. 2.

For $(p_a, p_b) \in B_1$ the solution is given as in (i), for $(p_a, p_b) \in B_2$ as in (ii), for $(p_a, p_b) \in B_3$ as in (iii) for $(p_a, p_b) \in B_4$ as in (iv).

## References

1. Cz. WOŹNIAK, *Materials reinforced by system of cords with constrained tensions*, J. Techn. Phys.

Резюме

УПРУГАЯ ПЛАСТИНА АРМИРОВАНА ГИБКИМИ ВОЛОКНАМИ

Используя результаты полученные в [1] выведено основную систему отношений для упругой пластины армированой гибкими валокнами. Даются точные решение плоской осево-симметрической граничной задачи.

Streszczenie

TARCZA SPRĘŻYSTA WZMOCNIONA UKŁADEM WIOTKICH CIĘGIEN

Korzystając z rezultatów uzyskanych w pracy [1] wprowadzono podstawowy układ związków lokalnych dla tarczy sprężystej wzmocnionej ukladem wiotkich włókien. Rozwiązano osiowo-symetryczne zagadnienie brzegowe dla pierścieniowej tarczy sprężystej wzmocnionej układem $n$ włókien rozmieszczonych koncentrycznie.

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