MECHANIKA
TEORETYCZNA
I STOSOWANA
1-2, 28 (1990)

SHAKEDOWN ANALYSIS IN THE CASE OF IMPOSED DISPLACEMENT

JAN A. KÖNIG, SŁAWOMIR PYCKO (WARSZAWA)

1. Introduction

The classical shakedown analysis of elastic-plastic structures exposed to variable repeated loads and/or to temperature variations neglects the possibility of occurrence, also, of some kinematical external actions e.g. imposed boundary displacements varying within prescribed limits. Let us notice that such actions can not be directly transformed into "equivalent" statical boundary conditions. Therefore, e.g. Koiter (1960) limited himself to the case of rigid-body motions of some parts of the body boundary.

It is the aim of this work to enlighten the problem of shakedown analysis in the case of imposed, variable repeated displacements. Corresponding extensions of the static as well as of the kinematic fundamental shakedown limits for this case will be compared with those proper for "equivalent" static loads.

2. Basic assumptions

We assume that the material of a given structure obeys the elastic, perfectly plastic model i.e.

$$\varepsilon_{ij} = \varepsilon^{E}_{ij} + \varepsilon^{p}_{ij} \,, \tag{2.1}$$

$$\varepsilon^{E}_{ij} = E^{-1}_{ijkl}\, \sigma_{kl} \,, \tag{2.2}$$

$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}\, \partial f / \partial \sigma_{ij}, \qquad \dot{\lambda} \geq 0 , \tag{2.3}$$

$$f(\sigma_{ij}) \leq k , \quad \dot{\lambda} f = 0 . \tag{2.4}$$

Here E_{ijkl} is the rank four elasticity tensor, ε_{ij}, ε^{E}_{ij}, ε^{p}_{ij} denote the total, elastic and plastic strain, respectively; σ_{ij} is the stress tensor and f( ) is a convex scalar-valued function of the stress. Thus if $f(\sigma_{ij}) \leq k$ and $f(\sigma^{o}_{ij}) \leq k$ then

$$(\sigma_{ij} - \sigma^{o}_{ij})\, \dot{\varepsilon}^{p}_{ij} \geq 0, \tag{2.5}$$

where ε^{p}_{ij} is associated with σ_{ij} via (2.3).

Strains ε_{ij} and displacements u_i are assumed to be sufficiently small so that geometrical linearity of the structural response might be hold.

The total stress and strain can be decomposed in the following way

$$\begin{aligned} \sigma_{ij} &= \sigma^{E}_{ij} + \rho_{ij} , \\ \varepsilon_{ij} &= E^{-1}_{ijkl}\sigma^{E}_{kl} + \varepsilon^{p}_{ij} + E^{-1}_{ijkl}\rho_{kl}, \end{aligned} \tag{2.6}$$

where σ^{E}_{ij}, called "elastic stress", is the stress calculated under the assumption of perfectly elastic structural response. Due to linearity of equilibrium equations, the part ρ_{ij} equilibrates vanishing loads.

Now, let us assume that a given structure of volume V bounded by surface S, is subjected to the following external actions:

1° Mechanical loads: body forces b within V and surface tractions t_i on a surface S_T;

2° Imposed displacements $\bar{u}_i$ on surface S_K;

3° The displacements u vanish on the remaining part S of the surface S.

The body forces b_i, surface tractions t_i and the imposed displacements $\bar{u}_i$ may vary arbitrary within some prescribed limits. These variations can be described, in the majority of practical cases, by means of a finite number of multipliers β_k:

$$b_i(x,t) = \sum \beta_k(t) b^{k}_i(x), \quad t_i(x,t) = \sum \beta_k(t) t^{k}_i(x), \quad \bar{u}_i(x,t) = \sum \beta_k(t) \bar{u}^{k}_i(x), \tag{2.7}$$

$$a_k \le \beta_k(t) \le b_k, \quad k = 1,\dots,r, \tag{2.8}$$

where

a_k, b_k being given constants, r number of independent sets of loads.

The elastic stress σ^E_{ij} is equal to

$$\sigma^E_{ij} = \sigma^{EM}_{ij} + \sigma^{EK}_{ij}, \tag{2.9}$$

where σ^{EM}_{ij} and σ^{EK}_{ij} are solutions of the following elasticity problems:

$$\begin{aligned}
&\sigma^{EM}_{ij,j} + b_i = 0, && \sigma^{EK}_{ij,j} = 0 && \text{in } V,\\
&\sigma^{EM}_{ij} n_j = t_i, && \sigma^{EK}_{ij} n_j = 0 && \text{on } S_T,\\
&u^{EM}_i = 0, && u^{EK}_i = 0 && \text{on } S_U,\\
&u^{EM}_i = 0, && u^{EK}_i = \bar{u}_i && \text{on } S_K,
\end{aligned} \tag{2.10}$$

where u^{EM}_i, u^{EK}_i are elastic displacements associated with the corresponding problems, n_i denotes the external unit vector normal to the surface S.

The residual stress ρ_{ij} appearing in the presence of plastic deformations obeys the following relations

$$\begin{aligned}
&\rho_{ij,j} = 0, && \varepsilon^P_{ij} + E^{-1}_{ijkl}\rho_{kl} = \tfrac{1}{2}(u^R_{i,j} + u^R_{j,i}) \quad \text{in } V,\\
&\rho_{ij}n_j = 0 && \text{on } S_T,\\
&u^R_i = 0, && \text{on } S_U + S_K,
\end{aligned} \tag{2.11}$$

whereas the total actual displacement is equal to

$$u_i = u^E_i + u^R_i = u^{EM}_i + u^{EK}_i + u^R_i. \tag{2.12}$$

3. The statical shakedown theorem

<u>Theorem</u> : if there exist a safety factor $s > 1$ and a statically admissible time-independent residual stress field $\bar{\rho}_{ij}(x)$

$$\bar{\rho}_{ij,j} = 0 \quad \text{in } V, \qquad \bar{\rho}_{ij} n_j = 0 \quad \text{on } S_T , \tag{3.1}$$

and such that for any combination of mechanical loads and imposed displacements possible to happen, the following conditions holds true

$$f[s(\sigma^{EM}_{ij}(x,t) + \sigma^{EK}_{ij}(x,t)) + \bar{\rho}_{ij}(x)] \le k , \tag{3.2}$$

then a given structure will shake down.

Proof: necessity of existence of the $\bar{\rho}_{ij}$ field is self-evident. To prove that it suffices for shakedown one can follow the classical proof by constructing the non-negative functional

$$L(t) = \frac{1}{2} \int_V E^{-1}_{ijkl} (\rho_{ij} - \bar{\rho}_{ij})(\rho_{kl} - \bar{\rho}_{kl}) \, dV \ge 0 , \tag{3.3}$$

It is easy to show, cf. e.g. Koiter (1960) that $\dot{L} \le 0$ and that the total plastic energy dissipated in an arbitrary long process is bounded, cf. e.g. König (1987),

$$W_p = \int_0^t \int_V \sigma_{ij} \dot{\varepsilon}^p_{ij} \, dV \, dt \le \frac{1}{2} \frac{s}{s-1} [L(0) - L(t)] \le$$
$$\le \frac{s}{2(s-1)} \int_V E^{-1}_{ijkl} (\bar{\rho}_{ij} - \rho^o_{ij})(\rho_{kl} - \rho^o_{kl}) \, dV , \tag{3.4}$$

where $\rho^o_{ij}(x) = \rho_{ij}(x,0)$ i.e. at $t = 0$.

4. The kinematical shakedown theorem

Theorem : if there exist, for a certain time interval (t_1, t_2):

(i) a history of body forces $b_i(x,t)$, surface tractions $t_i(x,t)$ and imposed displacements $\bar{u}_i(x,t)$ resulting in an elastic stress history $\sigma^E_{ij}(x,t) = \sigma^{EM}_{ij}(x,t) + \sigma^{EK}_{ij}(x,t)$,

(ii) a history of plastic strain field $\tilde{\epsilon}_{ij}(x,t)$ resulting in a kinematically admissible increment:

$$\Delta\tilde{\epsilon}_{ij}(x) = \tilde{\epsilon}_{ij}(x,t_2) - \tilde{\epsilon}_{ij}(x,t_1) = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i}) \quad \text{in } V,$$

$$\tilde{u}_i = 0 \quad \text{on } S_U + S_K, \tag{4.1}$$

so that the following inequality holds

$$\int_{t_1}^{t_2}\int_V (\sigma^{EM}_{ij}(x,t) + \sigma^{EK}_{ij}(x,t))\,\dot{\tilde{\epsilon}}_{ij}(x,t)\,dV\,dt > \int_{t_1}^{t_2}\int_V D(\dot{\tilde{\epsilon}}_{ij})\,dV\,dt, \tag{4.2}$$

then the body may not shake down. The symbol $D(\dot{\tilde{\epsilon}}_{ij})$ denotes the plastic energy rate associated uniquely with the $\dot{\tilde{\epsilon}}_{ij}$.

Proof: via "reductio ad absurdum" by assuming that a residual stress field $\bar{\rho}_{ij}(x)$ exists satisfying (3.2) for s=1 then obviously

$$[\tilde{\sigma}_{ij} - (\sigma^{EM}_{ij} + \sigma^{EK}_{ij} + \bar{\rho}_{ij})]\,\dot{\tilde{\epsilon}}_{ij} \geq 0, \tag{4.3}$$

where $\tilde{\sigma}_{ij}$ is defined by $D(\dot{\tilde{\epsilon}}_{ij}) = \tilde{\sigma}_{ij}\dot{\tilde{\epsilon}}_{ij}$.

By integrating (4.3) over the body volume and over the time interval (t_1, t_2) we arrive at

$$\int_{t_1}^{t_2}\int_V (\sigma^{EM}_{ij}(x,t) + \sigma^{EK}_{ij}(x,t))\,\dot{\tilde{\epsilon}}_{ij}(x,t)\,dV\,dt \leq \int_{t_1}^{t_2}\int_V D(\dot{\tilde{\epsilon}}_{ij})\,dV\,dt, \tag{4.4}$$

what contradicts the assumption (4.2).

Remark : the above-presented proofs are formally identical with those proper for the classical case. Therefore, the classical conclusion, cf. König (1987), concerning the separate criteria of incremental collapse and alternating plasticity are also valid in the case considered.

5. Equivalent load

Let us imagine that a given structure is subjected to variable repeated body forces b_i, surface tractions t_i on S_T and surface tractions

$t_i^{eq} = \sigma_{ij}^{EK} n_j$ acting on S_K. Let u_i equal zero on S_U. These external actions are called together "equivalent" load. A question arises whether shakedown limits in such a case are the same as in the case of imposed displacements. To answer this question let us notice that in the case of "equivalent load" the elastic stress is equal to

$$\sigma_{ij}^{E} = \sigma_{ij}^{EM} + \sigma_{ij}^{EK} + \hat{\sigma}_{ij}^{EM} , \tag{5.1}$$

where σ_{ij}^{EM} and σ_{ij}^{EK} are defined by (2.[illegible]) whereas $\hat{\sigma}_{ij}^{EM}$ follows from the following elasticity problem

$$\begin{array}{lll}
(\sigma^{EM} + \hat{\sigma}^{EM}) + b = 0 & \sigma^{EK} = 0 & \text{in } V , \\
(\sigma^{EM} + \hat{\sigma}^{EM})n = t & \sigma^{EK}n = 0 & \text{on } S , \\
(\sigma_{ij}^{EM} + \hat{\sigma}_{ij}^{EM})n_j = 0 & \sigma_{ij}^{EK}n_j = t^{eq} & \text{on } S_K , \\
u_i^{EM} + \hat{u}_i^{EM} = 0 & u_i^{EK} = 0 & \text{on } S_U .
\end{array} \tag{5.2}$$

Let us define a new boundary surface $\hat{S}_T = S_T + S_K$. In this case the shakedown problem is recognized as the classical one.

In view of the Melan static shakedown theorem, the shakedown conditions would be read in this case:

there must exist a time-independent residual stress field $\hat{\rho}_{ij}(x)$ so that the following relations hold true:

$$\begin{array}{l}
\hat{\rho}_{ij,j} = 0 \quad \text{in } V, \qquad \hat{\rho}_{ij}n_j = 0 \quad \text{on} \quad S_T + S_K , \\
f\ [\ s\ (\sigma_{ij}^{EM}(x,t) + \sigma_{ij}^{EK}(x,t) + \hat{\sigma}_{ij}^{EM}(x,t)) + \hat{\rho}_{ij}(x))\ ] \le k .
\end{array} \tag{5.3}$$

One can easily see that (5.3) differs from that condition formulated in section 3.

On the other hand, if considering this problem in the kinematical formulation, cf. section 4, we see that in the case of "equivalent" load the set of possible inadaptation modes (4.1) is greater than in that case of imposed displacements.

6. Example

Let us consider a two-span continuous I-beam, Fig. 1a, subjected

1° to variable repeated concentrated load $P_1(t)$ and, to imposed displacement $u_3(t)$, Fig. 1b, so that

$$0 \le P_1(t) \le \bar{P}_1 , \qquad 0 \le u_3(t) \le \bar{u}_3 , \tag{6.1}$$

where $\bar{P}_1$ and $\bar{u}_3$ are given constants,

2° to variable repeated concentrated loads $P_1(t)$, $P_3(t)$, Fig. 1c, independent of each other so that

$$0 \le P_1(t) \le \bar{P}_1 , \qquad 0 \le P_3(t) \le \bar{P}_3 . \tag{6.2}$$

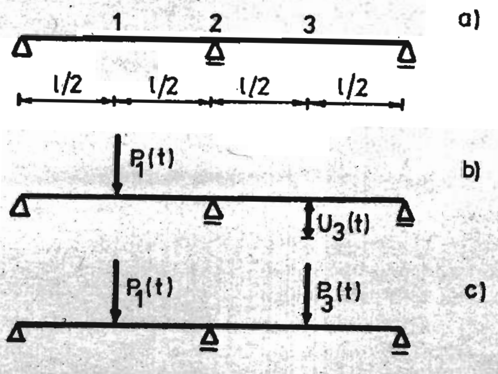

Fig. 1.

Let the magnitude of $\bar{P}_3$ be related to $\bar{u}_3$ so that, in the case of perfectly elastic response of the beam, the bending moments resulting from external actions $\bar{u}_3(t)$, $\bar{P}_3(t)$ are equal to each other. It takes place if

$$\bar{u}_3 = \frac{23}{1536} \bar{P}_3 l^3 /(EJ) , \tag{6.3}$$

where l is the span length, J inertia moment of the beam cross-section and E is Young's modulus.

In Fig. 2 there are shown the results of the shakedown analysis for the case 1° performed in accordance with theorems presented above (sections 3,4) and for the case 2° obtained by means of the classical analysis. The difference between the shakedown domains results from the smaller number of inadaptation modes existing in the case of imposed displacements (case 1°) in comparison with the case of "equivalent" load (case 2°).

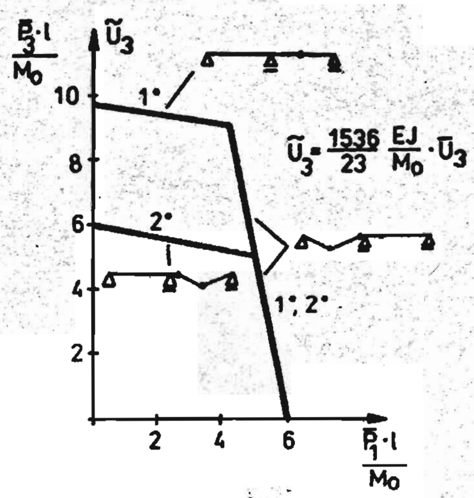

Fig. 2. Shakedown domains and modes of inadaptation.

References

Koiter, W.T. (1960): General theorems for elastic-plastic solids. In: Progress in Solid Mechanics, North-Holland, Amsterdam, 165-221.

König, J.A. (1987): Shakedown of elastic-plastic structures. Elsevier, Amsterdam-Oxford-New York-Tokyo and PWN, Warszawa

Summary

WYMUSZENIA KINEMATYCZNE W ANALIZIE PRZYSTOSOWANIA

Przedstawiono rozszerzenie twierdzeń teorii przystosowania na przypadek występowania zmiennych w czasie wymuszeń kinematycznych na brzegu ciała. Porównano na przykładzie belki ciągłej powyższy przypadek obciążenia z przypadkiem "równoważnych" obciążeń statycznych .