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FRACTURE OF PLANE BUNDLE OF INTERACTING ELASTIC FIBERS. STATICS

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The plane set of N elastic fibres is investigated. Each fibre interacts with the two nearest neighbours, the interaction force being proportional to the difference of the displacements. The dispersion relation has been written in explicit form. The displacement has been written as the sum of displacements corresponding to the subsequent N modes. The equations of statics are written as the set of N linear algebraic equations. The displacements and stresses have been calculated and plotted against the distance for the case when one or two fibres are damaged in a bundle of five fibers, and for the damage of the boundary fiber in a bundle of 21 fibers.

1. Interaction between fibers

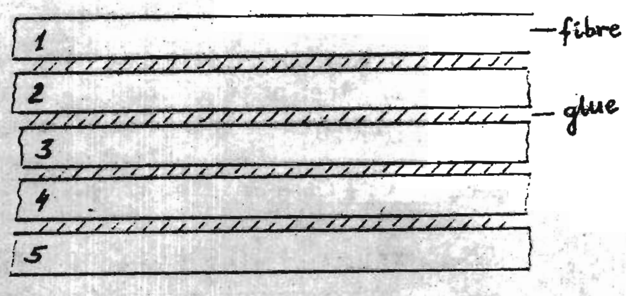

Fig. 1.

Consider N elastic fibers embedded in the elastic material, Fig.1. The displacement u_K of the fibre K is a function of the space coordinate x and time t. Each fiber is connected with two its neighbours by the elastic joints. The first and the last fibre is connected with one fibre only. We neglect the dynamics of the joints and assume that the force exerted by the fibre K on the fibre $K-1$ is proportional to the difference of the corresponding displacements $u_K - u_{K-1}$; the proportionality factor is denoted by H. The displacement u_K of the fiber K depends on the time t and the Cartesian space coordinate x

$$u_K = u_K(x,t) \tag{1.1}$$

In this paper we consider statics only and therefore the time dependence will be later eliminated. However, since the approach is parallel to that in the dynamic situations, (cf [1,2]) and since this does not consume any additional space we keep in the derivation the time dependence. The derived formulae will be used in the the dynamic solution, that will be discussed in a later paper.

N denotes the total number of fibers. The case $N = 1$ is trivial and was treated in many textbooks. More interesting is the interaction of one fibre with rigid surrounding treated by Sokolowski ([3]). One special form of displacement for $N = 2$ was treated in elsewhere (cf [4,5]). Here we assume $N \geq 2$. Each fiber has the same elastic modulus E and the same density ρ. The above assumptions lead to the following system of equations of motion

$$\begin{aligned}
&Eu_{1,xx} + H(u_2 - u_1) = \rho u_{1,tt}\\
&Eu_{2,xx} + H(u_3 + u_1 - 2u_2) = \rho u_{2,tt}\\
&Eu_{3,xx} + H(u_4 + u_2 - 2u_3) = \rho u_{3,tt}\\
&\dots\\
&Eu_{N-1,xx} + H(u_N + u_{N-2} - 2u_{N-1}) = \rho u_{N-1,tt}\\
&Eu_{N,xx} + H(u_{N-1} - u_N) = \rho u_{N,tt}
\end{aligned} \tag{1.2}$$

In the fibre K we expect two longitudinal harmonic waves, one of amplitude A_k running to the right and the other of amplitude B_K running to the left. Therefore the displacement u_K and stress σ in the fibre K are

$$\begin{aligned}
u_K &= A_K \exp i(-kx + \omega t) + B_K \exp i(kx + \omega t)\\
\sigma_K &= -ikEA_K \exp i(-kx + \omega t) + ikEB_K \exp i(kx + \omega t)
\end{aligned} \tag{1.3}$$

where ω is the frequency, and k the wave number.

Eqs (1.2) are satisfied, provided the amplitudes satisfy the system of algebraic equations

$$\begin{aligned}
(2p+1)A_1 &= A_2\\
(2p+2)A_2 &= A_3 + A_1\\
(2p+2)A_3 &= A_4 + A_2\\
(2p+2)A_4 &= A_5 + A_3
\end{aligned} \tag{1.4}$$

$$\dots$$
$$(2p+2)A_{N-1} = A_N + A_{N-2}$$
$$(2p+1)A_N = A_{N-1}$$

where
$$p = \frac{Ek^2 - \rho\omega^2}{2H} \tag{1.5}$$

Note other structure of the first and the last equations. Exactly the same system of equations holds for B_K, $K = 1,2,...,N$. In order to save space we do not write this system, remembering that everywhere in this and next chapter A may be replaced by B. The above system of equations may be written in the more convenient matrix form

$$\begin{bmatrix} 2p+1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2p+2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2p+2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2p+2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2p+2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2p+1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ \vdots \\ A_{N-1} \\ A_N \end{bmatrix} = 0 \tag{1.6}$$

Δ_N will denote the determinant of the coefficients

$$\Delta_N = \left.\begin{vmatrix} 2p+1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2p+2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2p+2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2p+2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2p+2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2p+1 \end{vmatrix}\right\} \begin{matrix} N \\ \text{rows} \end{matrix} \tag{1.7}$$

Define now the two closely related with Eq (1.7) $K \times K$ determinants, $2 \leq K \leq N$

$$R_K = \left.\begin{vmatrix} 2p+2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2p+2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2p+2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2p+2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2p+2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2p+1 \end{vmatrix}\right\} \begin{matrix} K \\ \text{rows} \end{matrix} \tag{1.8}$$

$$S_K = \begin{vmatrix} -1 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2p+2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2p+2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2p+2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 2p+2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2p+1 \end{vmatrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\end{matrix}}\right\} \begin{matrix} K \\ \text{rows} \end{matrix} \tag{1.9}$$

In order to calculate the determinants Δ_K and R_K first we express them by the determinants of lower order. There follow the two recursive relations

$$\Delta_K = (2p+1)R_{K-1} + S_{K-1} \tag{1.10}$$

$$R_K = (2p+2)R_{K-1} + S_{K-1} \tag{1.11}$$

Eliminate from the above relations the determinant S_{K-1}. There follow two recursive formulae for R_K and Δ_K

$$R_K = (2p+2)R_{K-1} - R_{K-2} \tag{1.12}$$

$$\Delta_K = (2p+1)R_{K-1} - R_{K-2} \tag{1.13}$$

These formulae allow one to calculate R_K and Δ_K as the functions of p for arbitrary K. However already for $K > 6$ this demands rather long calculations. In the next chapter introducing other description we shall be able to replace the recursive formulae by explicit functions. Here in order to gain some insight in the problem we give the first four determinants R_K and Δ_K expressed as the functions of p

$$\begin{aligned} R_1 &= 2p+1 \\ R_2 &= 4p^2+6p+1 \\ R_3 &= 8p^3+20p^2+12p+1 \\ R_4 &= 16p^4+56p^3+60p^2+20p+1 \\ &\dots \end{aligned} \tag{1.14}$$

$$\begin{aligned} \Delta_2 &= 4p(p+1) \\ \Delta_3 &= 2p(4p^2+8p+3) \\ \Delta_4 &= 8p(p+1)(2p^2+4p+1) \\ \Delta_5 &= 2p(4p^2+6p+1)(4p^2+10p+5) \\ &\dots \end{aligned} \tag{1.15}$$

Note that Δ_1 would correspond to a single fibre, not interacting with other fibers. We do not consider this degenerate case. Eq (1.6) posseses for each p the trivial solution $A_1 = A_2 = \ldots = A_N = 0$. It posseses the nontrivial solutions if the determinant Δ_N equals zero. Therefore p is not arbitrary, but must be calculated from the equation

$$\Delta_N = 0 \tag{1.16}$$

In order to gain some insight in the mechanics of the system we list here the roots of (1.16) for $N = 2,3,4,5$

$$\begin{aligned}
N = 2: &\quad p = 0,\ -1 \\
N = 3: &\quad p = 0,\ -\frac{1}{2},\ -\frac{3}{2} \\
N = 4: &\quad p = 0,\ -1,\ -1+\frac{1}{\sqrt{2}},\ -1-\frac{1}{\sqrt{2}} \\
N = 5: &\quad p = 0,\ \frac{-3-\sqrt{5}}{4},\ \frac{-3+\sqrt{5}}{4},\ \frac{-5-\sqrt{5}}{2},\ \frac{-5+\sqrt{5}}{2}
\end{aligned} \tag{1.17}$$

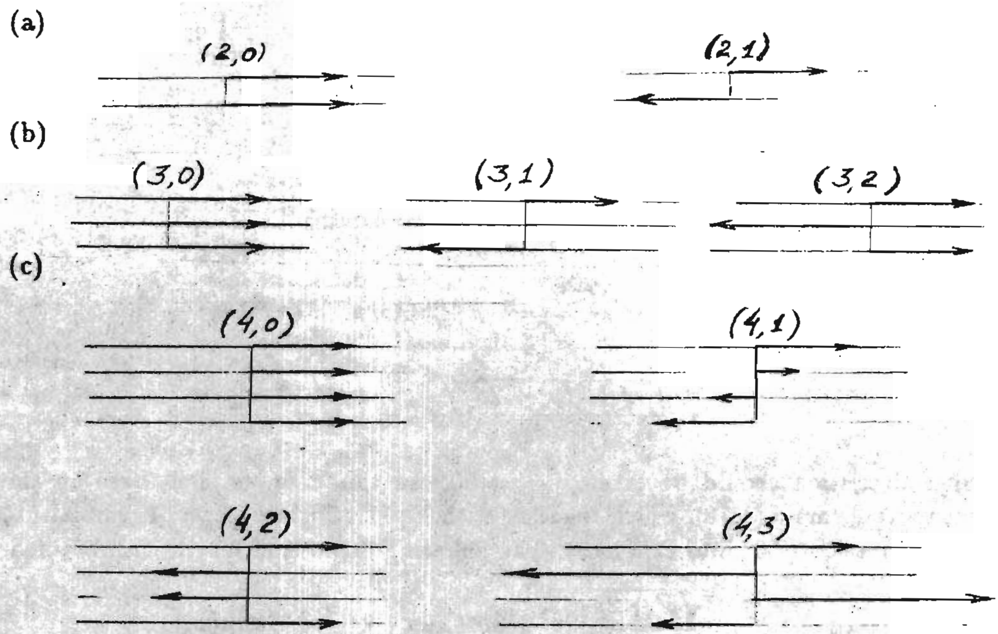

Fig. 2.

For each number of fibers one of the roots is $p = 0$. In accord with (1.8) this corresponds to $A_1 = A_2 = A_3 = A_4 = \ldots$ For $N = 2$ the second root $p = -1$ corresponds to $A_2 = -A_1$, (Fig.2a). The arrows represent the amplitudes or, equivalently, the displacement in the fibers at fixed time. For $N = 3$ the root $p = -1/2$ corresponds to $A_3 = -A_1$, $A_2 = 0$, and the root $p = -3/2$ corresponds to. $A_3 = A_1$, $A_2 = -2A_1$, (Fig.2b). For $N = 4$ the root $p = -1$ corresponds to $A_4 = A_1$, $A_2 = A_3 = -A_1$. To the root $p = -1 + 1/\sqrt{2}$ corresponds $A_2 = (-1+\sqrt{2})A_1$, $A_3 = (1-\sqrt{2})A_1$, $A_4 = -A_1$. To the root $p = -1 - 1/\sqrt{2}$ corresponds $A_4 = A_1$, $A_2 = A_3 = -(1+\sqrt{2})A_1$, (Fig.2c). Note that the number of roots and equivalently the number of different modes increases with the number of fibers and equals N.

2. Explicit functions

The form of the recursive formula (1.12) leads to the conclusion that the subsequent determinants R_K constitute the generalized Fibonacci numbers and may be represented by the formula

$$R_K = b_1\nu_1^K + b_2\nu_2^K \tag{2.1}$$

where b_1, b_2 are constants and ν_1, ν_2 are the solutions to the quadratic equation

$$\nu^2 - (2p+2)\nu + 1 = 0$$

equal to

$$\nu_{1,2} = p + 1 \pm \sqrt{p^2 + 2p} \tag{2.2}$$

The solution (2.1) must meet R_1, R_2 as given by Eq (1.14). The calculations lead to the following formula for R_K

$$\begin{aligned} R^K \;&=\; \frac{p+\sqrt{p^2+2p}}{\sqrt{p^2+2p}}(p+1+\sqrt{p^2+2p})^K + \\ &+\; \frac{-p+\sqrt{p^2+2p}}{\sqrt{p^2+2p}}(p+1-\sqrt{p^2+2p})^K \end{aligned} \tag{2.3}$$

Special caution should be taken if $p = 0$. In this case we shall base on the recursive formula (1.12), which together with Eq (1.14) gives $R_K = 1$. Substitute the expression (2.3) into Eq (1.15). The following expression for Δ_N is obtained

$$\begin{aligned} \Delta_N \;&=\; \frac{p+\sqrt{p^2+2p}}{2\sqrt{p^2+2p}}\left[2p^2+3p+(2p+1)\sqrt{p^2+2p}\right](p+1+ \\ &+\; \sqrt{p^2+2p})^{N-2} + \frac{-p+\sqrt{p^2+2p}}{2\sqrt{p^2+2p}}\left[2p^2+3p- \right. \\ &-\; \left.(2p+1)\sqrt{p^2+2p}\right](p+1-\sqrt{p^2+2p})^{N-2} \end{aligned} \tag{2.4}$$

Again the case $p = 0$ is not included in the above formula. The nontrivial solutions of the system (1.6) do exist, if the determinant equals zero. The analysis of Eq (2.4) leads to the conclusion that $\Delta_N = 0$ for a discrete set of values from the interval $-2 \leq p < 0$. From Eqs (1.10) and (1.15) it follows that for $p = 0$ there is $\Delta_N = 0$, too. Note that for $p < 0$ the values of the square root in Eqs (2.3) and (2.4) are imaginary.

Further we calculate the roots of Eq (2.4). Here we assume that p has already been calculated and calculate the amplitudes A_K for fixed N. Because the amplitudes are determined exact to in the multiplicative constant assume for simplicity $A_1 = 1$. In the next chapter when considering already the full system of amplitudes A_K, B_K we denote $A_1 = A$, $B_1 = B$ allowing arbitrary A, B.

In accordance with the matrix equation (1.6) we obtain the relation $A_2 = -(2p+1)A_1$, and the recursive formula

$$A_K = (2p+2)A_{K-1} - A_{K-2} \tag{2.5}$$

valid for $K > 2$. If we additionally assume $A_0 = 1$ then Eq (2.5) is valid for $K > 1$. This formula exactly coincides with the formula (1.12) for R_N. Therefore the calculations exactly parallel to that leading to Eq (2.1) give

$$A_K = c_1\nu_1^{K-1} + c_2\nu_2^{K-1}$$

where c_1, c_2 are constants. We must now take into account that $A_1 = 1$, $A_2 = 2p+1$ and calculate c_1, c_2. There is

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1\nu_1 + c_2\nu_2 &= 2p+1 \end{aligned}$$

and after solving for c_1, c_2 we obtain

$$\begin{aligned} A_K &= \frac{1}{2}\frac{p+\sqrt{p^2+2p}}{\sqrt{p^2+2p}}(1+p+\sqrt{p^2+2p})^{K-1} + \\ &+ \frac{1}{2}\frac{-p+\sqrt{p^2+2p}}{\sqrt{p^2+2p}}(1+p-\sqrt{p^2+2p})^{K-1} \end{aligned} \tag{2.6}$$

Note that for $p = 0$ all the amplitudes are equal $A_K =$ const. and there exists no relative motion of the fibers. Because of the assumption $A_1 = 1$ the above formula gives in fact only the ratios of the amplitudes.

It was mentioned above that the negative values of p are essential for the further calculations. For these values the roots in the above expressions are complex. Because of this we prefer to introduce the new parameter

$$s = -p \tag{2.7}$$

In order to simplify at this stage the notation define

$$R = \sqrt{2s - s^2}$$

(R is real for $-2 < p < 0$) and obtain from Eq (2.4)

$$\begin{aligned} \Delta_N &= \frac{1}{2\mathrm{i}R}\Big[-4s^3 + 8s^2 - 2s + 4\mathrm{i}(s^2 - s)R\Big](1 - s + \mathrm{i}R)^{N-2} - \\ &- \frac{1}{2\mathrm{i}R}\Big[-4s^3 + 8s^2 - 2s - 4\mathrm{i}(s^2 - s)R\Big](1 - s - \mathrm{i}R)^{N-2} \end{aligned} \tag{2.8}$$

Note that there hold the identities

$$\begin{aligned} -4s^3 + 8s^2 - 2s + 4\mathrm{i}(s^2 - s)R &= -2s(1 - s + \mathrm{i}R)^2 \\ -4s^3 + 8s^2 - 2s - 4\mathrm{i}(s^2 - s)R &= -2s(1 - s - \mathrm{i}R)^2 \end{aligned}$$

They allow one to write Eq (2.8) in the simple form

$$\Delta_N = \frac{-s}{\mathrm{i}R}(1 - s + \mathrm{i}R)^N + \frac{s}{\mathrm{i}R}(1 - s - \mathrm{i}R)^N \tag{2.9}$$

Introduce the parameter φ defined by the following relations

$$\begin{aligned} \varphi &= \arctan\frac{\sqrt{2s - s^2}}{1 - s} && \text{if } \ 0 \leq s < 1 \\ \varphi &= \arctan\frac{\sqrt{2s - s^2}}{1 - s} + \pi && \text{if } \ 1 \leq s < 2 \end{aligned} \tag{2.10}$$

Note that $0 \leq \varphi < \pi$ is single–valued continuous function of s for $0 < s < 2$. There hold the identities

$$\begin{aligned} (1 - s + \mathrm{i}R) &= \exp(\mathrm{i}\varphi) \\ (1 - s - \mathrm{i}R) &= \exp(-\mathrm{i}\varphi) \\ 1 - s &= \cos\varphi \\ \sqrt{2s - s^2} &= \sin\varphi \end{aligned} \tag{2.11}$$

in which s is the independent variable. If the independent variable is φ then Eq (2.11) uniquely defines $s(\varphi)$. The real–valued complex function (2.9) reduces now to the real function

$$\Delta_N = \frac{-2s}{\sqrt{2s - s^2}}\sin N\varphi = \frac{-2(1 - \cos\varphi)}{\sin\varphi}\sin N\varphi \tag{2.12}$$

This determinant must be equal zero. It follows immediately that the possible values of the parameter φ are defined by the relation $\sin N\varphi = 0$ which leads to the following expression for φ

$$\varphi = m\frac{\pi}{N} \qquad m = 0,1,2,3,...,N-1 \tag{2.13}$$

The integer m defines the mode of the solution. Obviously the modes $m \geq N$ do not introduce new functions. The above relation in accordance with Eqs (1.5), (2.10) and (2.7) defines the dispersion relation $k(\omega)$ for each mode m. In the next chapter we write this function in explicit form.

Transform now the formula (2.6) using the notation of Eqs (2.7), (2.9). We have $\nu_1 = \exp(\mathrm{i}\varphi)$, $\nu_2 = \exp(-\mathrm{i}\varphi)$ and the following formula is obtained

$$A_K = \cos(K-1)\varphi - \frac{s}{R}\sin(K-1)\varphi \tag{2.14}$$

The other possible forms may be obtained using Eq (2.11). There is

$$\begin{aligned} A_K &= \frac{1}{\sin\varphi}[\sin\varphi\cos(K-1)\varphi - (1-\cos\varphi)\sin(K-1)\varphi] = \\ &= \frac{1}{\sin\varphi}\sin K\varphi - \frac{1}{\sin\varphi}\sin(K-1)\varphi \end{aligned} \tag{2.15}$$

Note that for $K=1$ there is $A_1 = 1$. For $K = N$ in accordance with the second form of Eq (2.15) there is

$$A_N = \frac{1}{\sin\varphi}\sin N\varphi - \frac{1}{\sin\varphi}\sin(N-1)\varphi \tag{2.16}$$

The first term equals zero due to Eq (2.13). Due to the same equation and Eq (2.11) the second term equals either -1 or 1. For $j = 1,3,5,...$ there is $A_N = -1$, and for $j = 2,4,6,...$ there is $A_N = +1$. There follows the important qualitative result that in one mode the amplitudes of the two extreme fibers have equal moduli, and the same, or opposite signs.

3. The damaged bundle

Summarize the results derived above. For the mode m we have

$$\varphi^{(m)} = m\frac{\pi}{N} \qquad m = 0,1,2,...,N-1 \tag{3.1}$$

We consider now $\varphi^{(m)}$ as the independent variables and will express s and its functions as the functions of $\varphi^{(m)}$. In accordance with Eq (2.10) there holds the relation $(1-s)\tan\varphi^{(m)} = (2s-s^2)^{\frac{1}{2}}$. Solving this equation for s we obtain

$$s^{(m)} = 1 \pm \cos\varphi^{(m)} \tag{3.2}$$

For each $s^{(m)}$ the same sign must be taken. Remembering Eq (2.11) we infer that the $-$sign must be taken. There is

$$s^{(m)} = 1 - \cos m\frac{\pi}{N} \qquad \text{for } 0 \le m < N-1 \tag{3.3}$$

If the mode is prescribed, then the value of s for which $\Delta_N = 0$ is given by the above formulae. The dispersion relation for the mode m follows from Eq (1.4) and the relation $s = -p$

$$k = \mathrm{i}\sqrt{\frac{2H}{E}}\sqrt{s + \frac{\rho\omega^2}{2H}} \tag{3.4}$$

When deriving the relations governing the amplitudes A_1, A_2, A_3,...in the fibres 1,2,3,... in order to simplify the notation it was assumed $A_1 = 1$. In fact, the amplitude A_1 is in general different from 1, and moreover it is different for different modes. The amplitude of the wave of mode m running in the K fibre in the $+x$ direction will be denoted by $A_K^{(m)}$. $B_K^{(m)}$ will denote the amplitude of the wave of mode m running in the K fibre in the $-x$ direction. We denote now $A_1^{(m)} = A^{(m)}$, $B_1^{(m)} = B^{(m)}$ and obtain from Eq (2.16) the amplitudes for the mode $m > 0$ in the fibre K expressed by the two parameters $A^{(m)}$, $B^{(m)}$

$$A_K^{(m)} = A^{(m)}\frac{1}{\sin\varphi^{(m)}}[\sin K\varphi^{(m)} - \sin(K-1)\varphi^{(m)}] \tag{3.5}$$

$$B_K^{(m)} = B^{(m)}\frac{1}{\sin\varphi^{(m)}}[\sin K\varphi^{(m)} - \sin(K-1)\varphi^{(m)}] \tag{3.6}$$

For the mode $m = 0$ we have

$$A_K^{(0)} = A^{(0)} \qquad\qquad B_K^{(0)} = B^{(0)} \tag{3.7}$$

In the other paper we shall consider dynamics of the bundle and will use the full system of equations. In this paper we intend to consider only statics of the damaged bundle, and therefore in Eq (3.4) we assume $\omega = 0$. In this case the dependence on time drops out from all equations.

In accordance with Eqs (3.1) and (3.4) there is

$$k^{(m)} = \sqrt{-\frac{2H}{E}s^{(m)}}$$

Taking into account Eq (3.3) we obtain

$$k^{(m)} = \mathrm{i}\sqrt{\frac{2H}{E}}\sqrt{1-\cos m\frac{\pi}{N}} \qquad \text{for } 0 \le m \le N-1 \tag{3.8}$$

Denote

$$Q = \sqrt{\frac{2H}{E}} \tag{3.9}$$

There follows the formula for the displacement u_K

$$\begin{aligned} u_K &= \sum_m A_K^{(m)} \exp\Big(+x\sqrt{\frac{2H}{E}}\sqrt{1-\cos m\frac{\pi}{N}}\Big) + \\ &+ \sum_m B_K^{(m)} \exp\Big(-x\sqrt{\frac{2H}{E}}\sqrt{1-\cos m\frac{\pi}{N}}\Big) \end{aligned} \tag{3.10}$$

The summation is over all modes m, from $m = 0$ to $m = N-1$. Note that for $x \to \infty$ the first term tends to infinity, and for $x \to -\infty$ the second term tends to infinity.

We shall consider the case when one or more fibers are broken at $x = 0$. Because of this discontinuity we shall construct the solution from the solution for $x > 0$ and the solution for $x < 0$ demanding the continuity of displacement and stress in the non–damaged fibers. In the damaged fibers the stress equals zero at $x = 0$, and in the non–damaged fibers the displacement equals zero at $x = 0$. The symmetry of the mechanical system allows us to concentrate on the solution for $x > 0$ only. Obviously the mechanics excludes the infinite growth of the stresses, therefore $A_K^{(m)} = 0$. Take now into account Eq (3.5) expressing $B_K^{(m)}$ as the function of $B^{(m)}$. There results the formula for the displacement and stress in the fibre K for $x > 0$

$$\begin{aligned} u_K &= B^{(0)} + \sum_m \frac{B^{(m)}}{\sin m\frac{\pi}{N}}\Big[\sin\frac{Km\pi}{N} - \sin\frac{(K-1)m\pi}{N}\Big] \cdot \\ &\cdot \exp\Big[-xQ\sqrt{1-\cos m\frac{\pi}{N}}\Big] \end{aligned} \tag{3.11}$$

$$\begin{aligned} \sigma_K &= -EQ\sum_m \sqrt{1-\cos m\frac{\pi}{N}}\frac{B^{(m)}}{\sin m\frac{\pi}{N}}\Big[\sin\frac{Km\pi}{N} - \sin\frac{(K-1)m\pi}{N}\Big] \cdot \\ &\cdot \exp\Big[-xQ\sqrt{1-\cos m\frac{\pi}{N}}\Big] \end{aligned} \tag{3.12}$$

The summation is over the modes $m = 1$ to $m = N - 1$. To the mode $m = 0$ there corresponds zero stress. The m constants $B^{(m)}$ must be calculated to satisfy the boundary conditions. We shall consider further several different situations corresponding to damage of one fibre, either at the boundary,or inside the bundle, and the damage of two fibers either distant or neighbouring.

In order to gain some insight in the calculations for arbitrary N consider first the simplest possible system of three fibers, $N = 3$. In accordance with Eqs (3.1) and (3.4) there is

$$\varphi^{(0)} = 0 \qquad \varphi^{(1)} = \frac{\pi}{3} \qquad \varphi^{(2)} = \frac{2\pi}{3} \tag{3.13}$$

$$s^{(0)} = 0 \qquad s^{(1)} = \frac{1}{2} \qquad s^{(2)} = \frac{3}{2} \tag{3.14}$$

$$k^{(0)} = 0 \qquad k^{(1)} = \mathrm{i}\sqrt{\frac{1}{2}}\sqrt{\frac{2H}{E}} \qquad k^{(2)} = \mathrm{i}\sqrt{\frac{3}{2}}\sqrt{\frac{2H}{E}} \tag{3.15}$$

Substitute Eq (3.14) into the expressions for displacement and stress (1.3). For the mode 0 we obtain the following displacments and stresses at $x = 0$

$$\begin{aligned} &B_1^{(0)} = B_2^{(0)} = B_3^{(0)} = B^{(0)} \\ &u_1^{(0)} = u_2^{(0)} = u_3^{(0)} = B^{(0)} \\ &\sigma_1^{(0)} = \sigma_2^{(0)} = \sigma_3^{(0)} = 0 \end{aligned} \tag{3.16}$$

For the remaining two modes we obtain the following relations

$$\begin{array}{lll} B_1^{(1)} = B^{(1)} & B_2^{(1)} = 0 & B_3^{(1)} = -B^{(1)} \\ u_1^{(1)} = B^{(1)} & u_2^{(1)} = 0 & u_3^{(1)} = B^{(1)} \\ \sigma_1^{(1)} = -G\sqrt{\frac{1}{2}}B^{(1)} & \sigma_2^{(1)} = 0 & \sigma_3^{(1)} = G\sqrt{\frac{1}{2}}B^{(1)} \end{array} \tag{3.17}$$

$$\begin{array}{lll} B_1^{(2)} = B^{(2)} & B_2^{(2)} = -2B^{(2)} & B_3^{(2)} = -B^{(2)} \\ u_1^{(2)} = B^{(2)} & u_2^{(2)} = -2B^{(2)} & u_3^{(2)} = B^{(2)} \\ \sigma_1^{(2)} = -G\sqrt{\frac{3}{2}}B^{(2)} & \sigma_2^{(2)} = 2G\sqrt{\frac{3}{2}}B^{(2)} & \sigma_3^{(2)} = -G\sqrt{\frac{3}{2}}B^{(2)} \end{array} \tag{3.18}$$

where

$$G = \sqrt{2EH} \tag{3.19}$$

Denote the stress in each fibre of the non–damaged, ideal bundle by σ. To this initial stress corresponds the initial displacement $u = x\sigma/E$. Assume that the fibre $k = 3$ has been broken at $x = 0$. The fibres $k = 1$ and $k = 2$ are non-damaged. At $x = 0$ the displacement in the fibres 1 and 2, and the total stress in the fibre 3 equal zero. The stress in the fibre 3 consists of the initial stress σ and the additional stress $\sigma_K^{(0)} + \sigma_K^{(1)} + \sigma_K^{(2)}$. In accordance with Eqs (3.16) $\div$ (3.18) for $x = 0$ there hold the relations

- displacement in the first fibre

$$B^{(0)} + B^{(1)} + B^{(2)} = 0$$

- displacement in the second fibre

$$B^{(0)} - 2B^{(2)} = 0$$

- stress in the third fibre

$$G\sqrt{\frac{1}{2}}B^{(1)} - G\sqrt{\frac{3}{2}}B^{(2)} + \sigma = 0$$

Note that at $x = 0$ the initial displacement equals zero and therefore it does not influence the above expressions. In the further calculations we assume $\sigma/G = 1$. Due to the linearity of the system for other values of σ/G the resulting displacements are σ/G times larger. We face the system of equations in the constants $B^{(K)}$

$$\begin{aligned} &B^{(0)} + B^{(1)} + B^{(2)} = 0 \\ &B^{(0)} - 2B^{(2)} = 0 \\ &\sqrt{\frac{1}{2}}B^{(1)} - \sqrt{\frac{3}{2}}B^{(2)} = -1 \end{aligned} \tag{3.20}$$

The solution of the above system of linear algebraic equations is

$$\begin{aligned} B^{(0)} &= \frac{2\sqrt{2}}{3+\sqrt{3}} \\ B^{(1)} &= -\frac{3\sqrt{2}}{3+\sqrt{3}} \\ B^{(2)} &= \frac{\sqrt{2}}{3+\sqrt{3}} \end{aligned} \tag{3.21}$$

We can now calculate the displacement corresponding to modes 0,1,2. The next step is the sumation over all modes and calculation of the total displacements $u_K(x)$ and the total stresses $\sigma_K(x)$ in the fibre K. The formulae (3.11) and (3.12) reduce to the expressions

$$\begin{aligned} u_1(x) &= \frac{\sigma}{E}x + B^{(0)} + B^{(1)}\exp\Big(-xQ\sqrt{\frac{1}{2}}\Big) + B^{(2)}\exp\Big(-xQ\sqrt{\frac{3}{2}}\Big) \\ u_2(x) &= \frac{\sigma}{E}x + B^{(0)} - 2B^{(2)}\exp\Big(-xQ\sqrt{\frac{3}{2}}\Big) \\ u_3(x) &= \frac{\sigma}{E}x + B^{(0)} - B^{(1)}\exp\Big(-xQ\sqrt{\frac{1}{2}}\Big) + B^{(2)}\exp\Big(-xQ\sqrt{\frac{3}{2}}\Big) \end{aligned} \tag{3.22}$$

$$\sigma_1(x) = \sigma - EQB^{(1)}\sqrt{\frac{1}{2}}\exp\left(-xQ\sqrt{\frac{1}{2}}\right) - EQB^{(2)}\sqrt{\frac{3}{2}}\exp\left(-xQ\sqrt{\frac{3}{2}}\right)$$
$$\sigma_2(x) = \sigma - EQB^{(2)}\sqrt{\frac{3}{2}}\exp\left(-xQ\sqrt{\frac{3}{2}}\right) \tag{3.23}$$
$$\sigma_3(x) = \sigma - EQB^{(1)}\sqrt{\frac{1}{2}}\exp\left(-xQ\sqrt{\frac{1}{2}}\right) - EQB^{(2)}\sqrt{\frac{3}{2}}\exp\left(-xQ\sqrt{\frac{3}{2}}\right)$$

The corresponding curves for $u_K(x)$ are given as solid lines in Fig.4. In order to obtain more clear plots in Fig.3 was omitted the initial displacement $x\sigma/E$. Note the displacement jump at $x = 0$ in the fibre 3. At large distance from $x = 0$ the displacements in all fibres are the same. The stresses $\sigma_K(x)$ are given as the broken lines in Fig.3. Note that for $x < 1$ there is $\sigma_2 > \sigma_1$ and for $x > 1.5$ there is $\sigma_2 < \sigma_1$.

4. Statics of N fibers

Consider now a bundle of N fibers. Consider first the damage at $x = 0$ of one fibre labelled $K = D$. There is

$$\begin{aligned} &\sigma_D = 0 \\ &u_1 = u_2 = ... = u_{D-1} = u_{D+1} = ... = u_N = 0 \\ &x = 0 \end{aligned} \tag{4.1}$$

From Eqs (3.9) and (3.10) there follows

$$\sum_m \sqrt{1 - \cos m\frac{\pi}{N}}\frac{B^{(m)}}{\sin m\frac{\pi}{N}}\left[\sin\frac{Dm\pi}{N} - \sin\frac{(D-1)m\pi}{N}\right] = 0 \tag{4.2}$$

$$B^{(0)} + \sum_m \frac{B^{(m)}}{\sin m\frac{\pi}{N}}\left[\sin\frac{Km\pi}{N} - \sin\frac{(K-1)m\pi}{N}\right] = 0$$
$$\text{for} \quad K \neq D \tag{4.3}$$

The summation is for $m = 1$ to $m = N - 1$. The above system of N algebraic equations allows one to calculate the N unknowns $B^{(m)}$, and they in turn determine the displacements in each fibre.

The heavy lines in Fig.4 give the displacements in each fibre as the function of x for the bundle of 5 fibers provided the first fibre has been damaged. The broken lines give the coresponding stresses as the function of x. Note that the stress in the fibre 2 for $x < 1$ is larger than in the other fibers. For $x > 2$ the stress in the fibre 2 is smaller than the stress in the fibres 3,4,5.

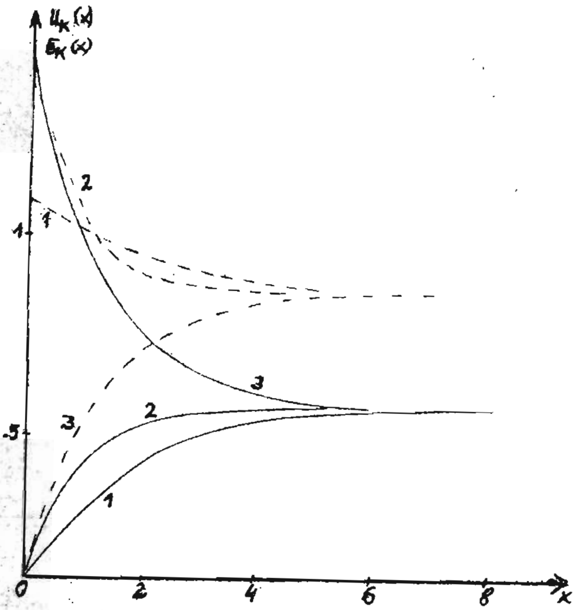
$u_K(x)$
$\sigma_K(x)$
1
.5
0
2
4
6
8
x
1
2
3
2
3
1

Fig. 3.

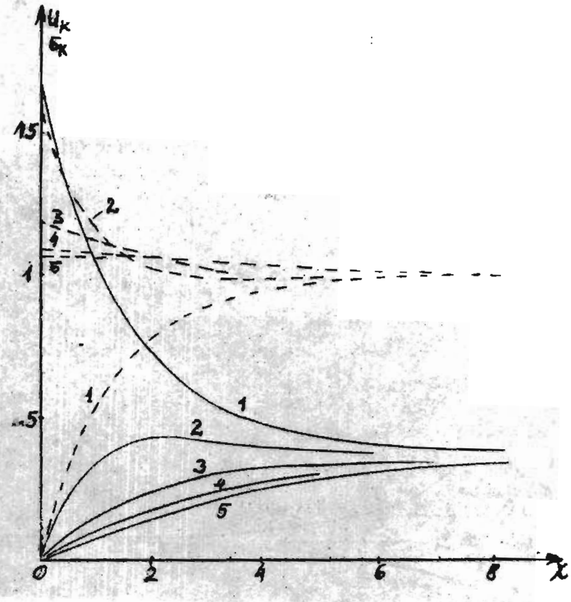
u_K
σ_K
1.5
1
.5
0
2
4
6
8
x
2
3
4
5
1
1
2
3
4
5

Fig. 4.

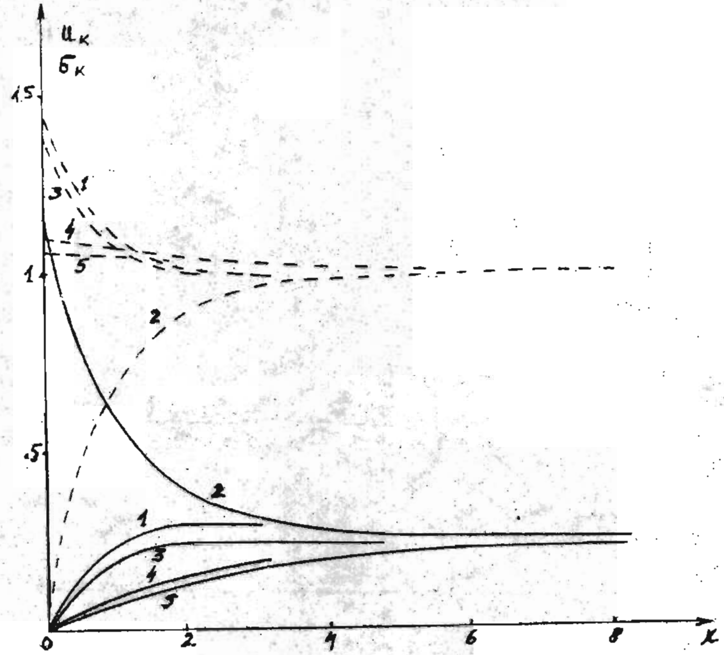

Fig. 5.

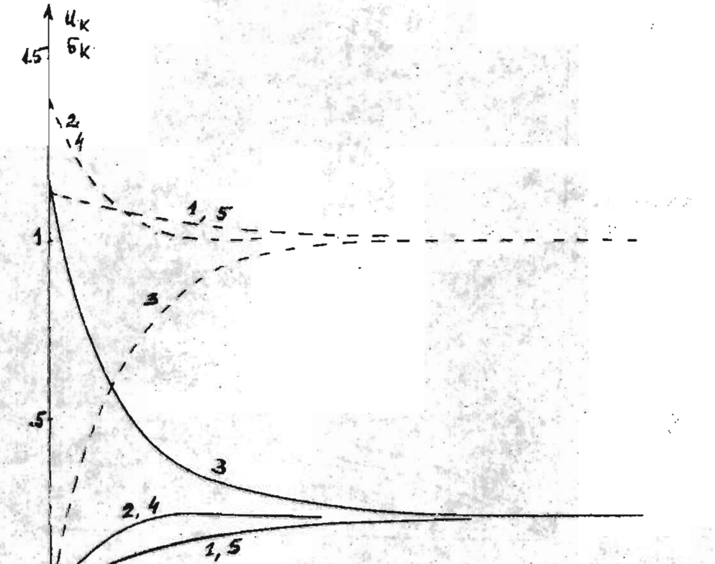

Fig. 6.

Fig.5 and Fig.6 give the displacement when the second or third fibers have been damaged. The corresponding stresses are shown as the broken lines. Note that at large distance the displacements and stresses in all fibers, damaged and not damaged are equal.

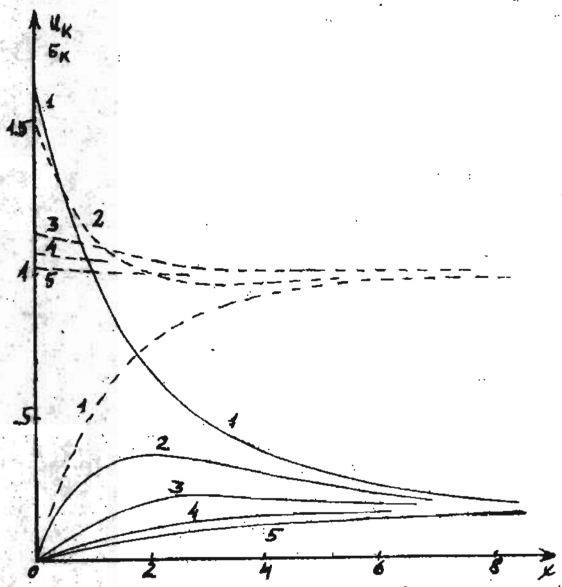

Fig. 7.

Very interesting is the case when the boundary fiber of a wide bundle is broken. To gain some insight we give the corresponding curves for 21 fibers (Fig.7). In all plots the initial displacement was omitted.

Finally we consider the case when two fibers labelled $K = D_1$ and $K = D_2$ have been damaged

$$\begin{aligned} &\sigma_{D_1} = 0 \qquad &&\sigma_{D_2} = 0 \qquad &&x = 0 \\ &u_K = 0 \qquad &&\text{for } K \neq D_1, D_2 \qquad &&x = 0 \end{aligned} \tag{4.4}$$

The above boundary conditions we write as the equations for $B^{(m)}$

$$\sum_m \sqrt{1 - \cos m\frac{\pi}{N}} \frac{B^{(m)}}{\sin m\frac{\pi}{N}} \left[\sin\frac{D_1 m\pi}{N} - \sin\frac{(D_1 - 1)m\pi}{N}\right] = 0 \tag{4.5}$$

$$\sum_m \sqrt{1 - \cos m\frac{\pi}{N}} \frac{B^{(m)}}{\sin m\frac{\pi}{N}} \left[\sin\frac{D_2 m\pi}{N} - \sin\frac{(D_2 - 1)m\pi}{N}\right] = 0 \tag{4.6}$$

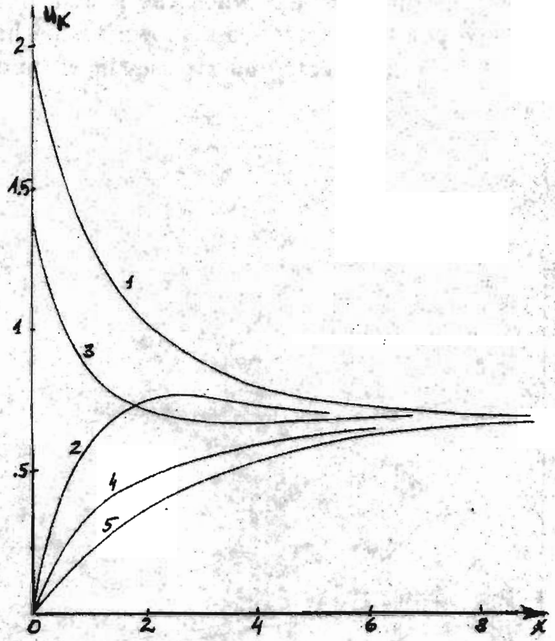

Fig. 8.

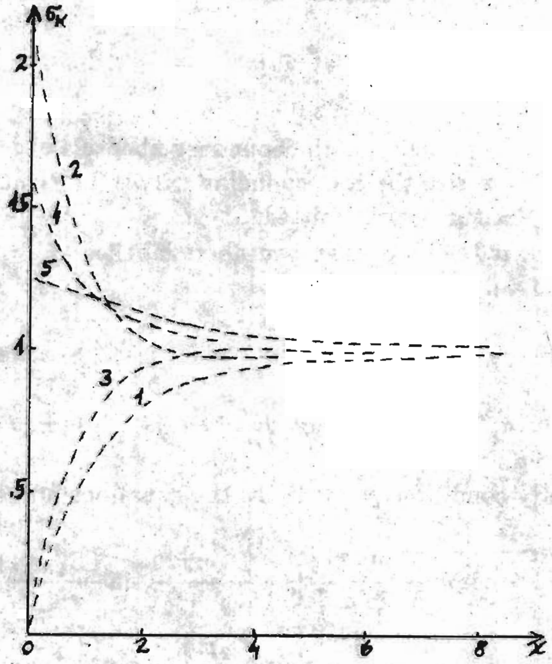

Fig. 9.

$$B^{(0)} + \sum_m \frac{B^{(m)}}{\sin m\frac{\pi}{N}} \Big[\sin\frac{Km\pi}{N} - \sin\frac{(K-1)m\pi}{N}\Big] = 0$$

$$\text{for} \quad K \neq D_1, D_2 \qquad (4.7)$$

Fig.8. gives the displacements in the bundle of five fibers, when the first and the third fibres have been damaged. Note that the displacement in the second (non-damaged) fibre in one interval of x is larger than the displacement in the third (damaged) fiber. The corresponding stresses are given in Fig.9. At large distances from $x = 0$ the stresses and displacements in all fibers are equal.

Generalisation of the above formulae to the damage of more than two fibers can be done by adding additional equations of the (4.5) type.

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Przełom płaskiej wiązki elastycznych włókien. Statyka

Streszczenie

W pracy badano płaską wiązkę N elastycznych włókien. Każde z nich oddziaływuje z siłą proporcjonalną do różnicy przemieszczeń na dwa sąsiednie włókna. Równanie dyspersyjne zostało przedstawione w postaci rozwiniętej. Przemieszczenie przedstawiono jako sumę przemieszczeń odpowiadających N kolejnym modom. Równanie statyki zapisano jako układ N liniowych równań algebraicznych. Naprężenia i przemieszczenia obliczono i przedstawiono w funkcji odległości. Przedstawiono 3 przypadki: zerwanie 1 lub 2 włókien w wiązce 5 włókien oraz zerwanie włókna brzegowego w wiązce 21 włókien.

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