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SOME COMENTS ON REPRESENTATION OF VECTOR-VALUED ISOTROPIC FUNCTION

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From the general scalar-valued isotropic function a complete and irreducible representation of a general vector-valued isotropic function of an arbitrary number of symmetric tensors, vectors and skew-symmetric tensors is invented.

1. Introduction

The complete irreducible representation of a vector-valued isotropic function (not necessarily polynomial), Smith (1971), Wang (1969), (1970) and (1971) is directly deducible from the minimal functional basis of a scalar-valued isotropic function a priori linear in one of the argument vectors. The minimal functional basis of a scalar-valued isotropic function (not necessarily polynomial) a priori linear in one of the argument vectors is directly obtainable from the general minimal functional basis, Boehler (1977) and (1987), Smith (1971), Wang (1969), (1970) and (1971).

In this paper a general representation of a vector-valued isotropic function of an arbitrary number of symmetric tensors, vectors and skew-symmetric tensors is deduced. The present method is similar to the method used by Korsgaard (1990) in order to determine the vector generators in the two-dimensional case. The obtained results coincide with those arrived at by Smith (1971) with the aid of different methods.

Throughout this paper tensor means second-order 3-dimensional tensor and vector means 3-dimensional vector.

2. Formulation of the problem

The requirement of form-invariance for a vector-valued function $\boldsymbol{f}$ of the symmetric tensors $\mathbf{A}_i$ $(i = 1, ..., N)$, the vectors $\boldsymbol{v}_m$ $(m = 1, ..., M)$ and the skew-symmetric tensors $\mathbf{W}_p$ $(p = 1, ..., P)$ under the full orthogonal grup $O(3)$ is expressed as follows

$$\boldsymbol{f}(\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathsf{T}}, \mathbf{Q}\boldsymbol{v}_m, \mathbf{Q}\mathbf{W}_p\mathbf{Q}^{\mathsf{T}}) = \mathbf{Q}\boldsymbol{f}(\mathbf{A}_i, \boldsymbol{v}_m, \mathbf{W}_p) \qquad \forall \mathbf{Q} \in O(3) \tag{2.1}$$

The vector-valued functions, which satisfy Eq (2.1), are called isotropic.

The requirement of form-invariance for a vector-valued isotropic function $\boldsymbol{f}(\mathbf{A}_i, \boldsymbol{v}_m, \mathbf{W}_p)$ is explicitly satisfied by introducing an auxiliary vector $\boldsymbol{d}$ and forming a scalar-valued isotropic function f equal to the scalar product between $\boldsymbol{f}$ and $\boldsymbol{d}$

$$f(\mathbf{A}_i, \boldsymbol{v}_m, \mathbf{W}_p, \boldsymbol{d}) = \boldsymbol{f} \cdot \boldsymbol{d} \tag{2.2}$$

The general representation theorem for scalar-valued isotropic functions states, Boehler (1977) and (1987), Smith (1971), Wang (1969), (1970) and (1971), that a function (2.2) can be expressed as a single-valued function of the invariants of the functional basis of the argument tensors and vectors. The functional basis for an arbitrary set of tensors and vectors under the full orthogonal grup is given in Table 1. The corrected list of invariants is obtained from Boehler (1977), (1987) and Smith (1971). The functional basis have been proven to be minimal by Pennisi and Trovato (1987).

Table 1. The functional basis for the full orthogonal group

$\mathrm{tr}\mathbf{A}_i$; $\mathrm{tr}\mathbf{A}_i^2$; $\mathrm{tr}\mathbf{A}_i^3$; $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j$; $\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j$; $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j^2$; $\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j^2$; $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j\mathbf{A}_k$	
$\boldsymbol{v}_m \cdot \boldsymbol{v}_m$; $\boldsymbol{v}_m \cdot \boldsymbol{v}_n$	$i, j, k = 1, ..., N$; $i < j < k$
$\mathrm{tr}\mathbf{W}_p^2$; $\mathrm{tr}\mathbf{W}_p\mathbf{W}_q$; $\mathrm{tr}\mathbf{W}_p\mathbf{W}_q\mathbf{W}_r$	$m, n = 1, ..., M$; $m < n$
$\boldsymbol{v}_m \cdot \mathbf{A}_i\boldsymbol{v}_m$; $\boldsymbol{v}_m \cdot \mathbf{A}_i^2\boldsymbol{v}_m$; $\mathbf{A}_i\boldsymbol{v}_m \cdot \mathbf{A}_j\boldsymbol{v}_m$	$p, q, r = 1, ..., P$; $p < q < r$
$\boldsymbol{v}_m \cdot \mathbf{A}_i\boldsymbol{v}_n$; $\boldsymbol{v}_m \cdot \mathbf{A}_i^2\boldsymbol{v}_n$; $\mathbf{A}_i\boldsymbol{v}_m \cdot \mathbf{A}_j\boldsymbol{v}_n - \mathbf{A}_i\boldsymbol{v}_n \cdot \mathbf{A}_j\boldsymbol{v}_m$	
$\boldsymbol{v}_m \cdot \mathbf{W}_p^2\boldsymbol{v}_m$; $\mathbf{W}_p\boldsymbol{v}_m \cdot \mathbf{W}_q\boldsymbol{v}_m$; $\mathbf{W}_p^2\boldsymbol{v}_m \cdot \mathbf{W}_q\boldsymbol{v}_m$; $\mathbf{W}_p\boldsymbol{v}_m \cdot \mathbf{W}_q^2\boldsymbol{v}_m$	
$\boldsymbol{v}_m \cdot \mathbf{W}_p\boldsymbol{v}_n$; $\boldsymbol{v}_m \cdot \mathbf{W}_p^2\boldsymbol{v}_n$; $\mathbf{W}_p\boldsymbol{v}_m \cdot \mathbf{W}_q\boldsymbol{v}_n - \mathbf{W}_p\boldsymbol{v}_n \cdot \mathbf{W}_q\boldsymbol{v}_m$	
$\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2$; $\mathrm{tr}\mathbf{A}_i^2\mathbf{W}_p^2$; $\mathrm{tr}\mathbf{A}_i^2\mathbf{W}_p^2\mathbf{A}_i\mathbf{W}_p$; $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q$; $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2\mathbf{W}_q$; $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p\mathbf{W}_q^2$	
$\mathrm{tr}\mathbf{A}_i\mathbf{A}_j\mathbf{W}_p$; $\mathrm{tr}\mathbf{A}_i^2\mathbf{A}_j\mathbf{W}_p$; $\mathrm{tr}\mathbf{A}_i\mathbf{A}_j^2\mathbf{W}_p$; $\mathrm{tr}\mathbf{A}_i\mathbf{W}_p^2\mathbf{A}_j\mathbf{W}_p$	
$\mathbf{A}_i\boldsymbol{v}_m \cdot \mathbf{W}_p\boldsymbol{v}_m$; $\mathbf{A}_i^2\boldsymbol{v}_m \cdot \mathbf{W}_p\boldsymbol{v}_m$; $\mathbf{A}_i\mathbf{W}_p\boldsymbol{v}_m \cdot \mathbf{W}_p^2\boldsymbol{v}_m$; $\mathbf{A}_i\boldsymbol{v}_m \cdot \mathbf{W}_p\boldsymbol{v}_n - \mathbf{A}_i\boldsymbol{v}_n \cdot \mathbf{W}_p\boldsymbol{v}_m$	

The scalar-valued isotropic function f given by (2.2) is a priori linear in $\boldsymbol{d}$. Thus f cannot contain invariants of higer order in $\boldsymbol{d}$ (the invariant $\boldsymbol{d} \cdot \boldsymbol{d}$ has to be excluded from the functional basis). Consequently the functional basis (Table 1) is directly applicable for the determination of the invariants containing $\boldsymbol{d}$. The set of invariants linear in $\boldsymbol{d}$, is given in Table 2.

Table 2. The invariants linear in d

$d \cdot v_m$	
$d \cdot A_i v_m$; $d \cdot A_i^2 v_m$	$i, j = 1, ..., N$; $i < j$
$d \cdot W_p v_m$; $d \cdot W_p^2 v_m$	$m = 1, ..., M$
$A_i d \cdot A_j v_m - A_i v_m \cdot A_j d$	$p, q = 1, ..., P$; $p < q$
$W_p d \cdot W_q v_m - W_p v_m \cdot W_q d$	
$A_i d \cdot W_p v_m - A_i v_m \cdot W_p d$	

The minimal functional basis of a scalar-valued function linear in one of the argument vectors contains fewer invariants then the minimal basis of a general scalar-valued isotropic function.

The representation of the scalar-valued isotropic function (2.2) becomes

$$f(A_i, v_m, W_p, d) = f(I_s, J_t) = \sum_{t=1}^{T} \Phi_t(I_s) J_t \tag{2.3}$$

where Φ_t $(t = 1, ..., T)$ are scalar-valued isotropic function of the invariants I_s $(s = 1, ..., S)$ of the functional basis (see Table 1), which do not contain the vector d and J_t $(t = 1, ..., T)$ are the invariants of the functional basis, which are linear in d (see Table 2).

3. The complete irreducible representation of vector-valued isotropic functions

By differentiating f, Eq (2.3), with respect to d a representation of the vector-valued isotropic function $f(A_i, v_m, W_p)$ is obtained

$$f(A_i, v_m, W_p) = \frac{\partial f}{\partial d} = \sum_{t=1}^{T} \Phi_t(I_s) \frac{\partial J_t}{\partial d} = \sum_{t=1}^{T} \Phi_t(I_s) g_t \tag{3.1}$$

where g_t are basic form-invariant vector-valued functions called generators. The representation of f given by Eq (3.1) is called canonical form of f. The generators g_t derived by the procedure described above are listed in Table 3.

The aforementioned list of generators is equal to that one obtained by Smith (1971). Smith followed the general otuline of the proof given by Wang (1969), (1970) and (1971). He noticed, Smith (1970) and (1971), that the results given by Wang could be sharpened, and consequently produced a list of generators smaller then that of Wang.

Table 3. The generators of a vector-valued isotropic function

v_m	$i,j = 1,...,N;\ i < j$
$\mathbf{A}_i v_m;\ \ \mathbf{A}_i^2 v_m$	$m = 1,...,M$
$\mathbf{W}_p v_m;\ \ \mathbf{W}_p^2 v_m$	$p,q = 1,...,P;\ p < q$
$(\mathbf{A}_i\mathbf{A}_j - \mathbf{A}_j\mathbf{A}_i)v_m$	
$(\mathbf{W}_p\mathbf{W}_q - \mathbf{W}_q\mathbf{W}_p)v_m$	
$(\mathbf{A}_i\mathbf{W}_p - \mathbf{W}_p\mathbf{A}_i)v_m$	

The list of generators (Table 3) is complete and the generators are irreducible, Pennisi and Trovato (1987), i.e. neither of the generators is expressible by the remaining ones. Obviously this does not mean that these vectors are linearly independent, but that in every null linear combination of them every coefficient is a scalar function assuming the zero value for some values of variables, Pennisi and Trovato (1987). The representations (3.1) means, that if there, at a given point, are $L \leq 3$ linearly independent vectors among g_t, then f is expressible as a linear combination of any L linearly independent vectors chosen from these. Thus the representations of vector-valued isotropic functions obtained by using Tables 1 and 3 are complete and irreducible.

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O reprezentacji izotropowej funkcji wektorowej

Streszczenie

Wykorzystując ogólną reprezentację skalarnej funkcji izotropowej wyprowadzono ogólną, kompletną i nieredukowalną reprezentację izotropowej funkcji wektorowej zależnej od dowolnej liczby symetrycznych i antysymetrycznych tensorów i wektorów.

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