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ANALISIS OF BENDING CURVE AND CRITICAL LOAD OF A VARIABLE CROSS-SECTION BEAM BY MEANS OF INFLUENCE FUNCTION METHOD

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In this paper the method of influence function is applied to solving of the problem of deflection curve and critical load finding in a variable cross-section beam. The function describing the deflection curve of an elastic supported beam with variable flexular rigidity is obtained in terms of the Cauchy function in a power series form. A general form of a characteristic equation is obtained which made it possible to calculate the estimators of critical Euler load. Some examples of different beams being used frequently in practice are given in details. The results are compared with the well-known theoretical results and show a good agreement.

1. Introduction

In many cases, like aerial masts, towers, outriggers, spindles of machine tods, turbine blades, a variable cross section of the structural elements as well as axial and shear loads should be correctly considered. However, because of numerous difficulties, the possible framework of such an analysis is limited and anly a few solutions to a problem of determining deflection curve of a variable cross-section beam have been found (cf Timoshenko and Guder, 1979; Zoryj, 1982). The similar situation arises in the problem of the stability of a beam subject to a conipressive Euler force (cf Timoshenko, 1971). Analytical and numerical methods (method of successive approximators, iterative variational, finite difference, finite element and transfer matrix methods) are most frequently used in such analyses. In the present paper a method of influence function is applied. The method has been used in the study on vibrations of flexural beams (cf Jaroszewicz and Zoryj, 1983, 1985). The method is based on the mathematical similarity of differential equations describing vibrations and deflection of beams, which are fourth order equations with variable coefficients.

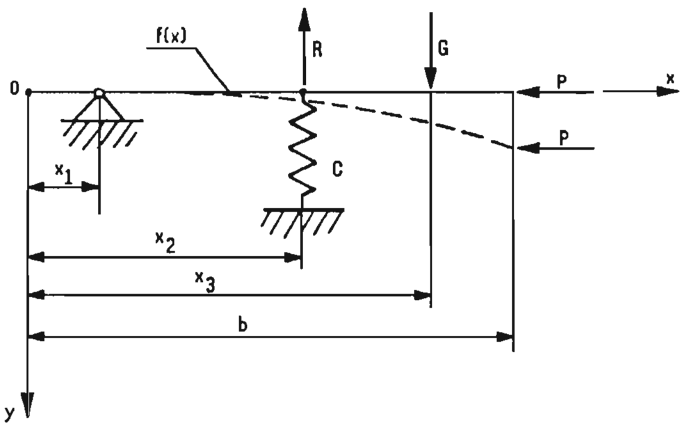

Fig. 1.

Fig.1 gives a model of elastically supported beam subject to a transverse load G and an axial compressive load P at the free end $(x = b)$. The rigidity of the support at $x = x_2$ is c. The flexural ridigity is $f = EJ(x)$, where the Young modulus of elasticity E =const as the material is assummed to be homogeneaus and the plane moment of interia $J(x)$ is variable. The function $1/f(x)$ shoud be continuous, positive definite and should have a finite value and integral in $[a, b]$.

2. Definition of the problem

The deflection of the model beam given in Fig.1 is defined as follows

$$L[y] = -R\delta(x - x_2) + G\delta(x - x_3) \tag{2.1}$$

where the differential operator $L[y] = (fy'')'' + fy''$, and δ is the Dirac function and R stands for the bearing reaction.

The boundary conditions are

$$
\begin{aligned}
& y(a) = 0 && (fy'')\Big|_{x=a} = 0 \\
& (fy'')\Big|_{x=b} = 0 && [(fy'') + Py']\Big|_{x=b} = 0
\end{aligned}
\tag{2.2}
$$

and the following can be written for the supported end

$$
cy(x_2) = R \tag{2.3}
$$

The following solution to the general equation (2.1) can be proposed (cf Zoryj, 1982)

$$
y = C_0 + C_1(x-a) + C_2K_{x1} + C_3\dot{K}_{x1} - R\Phi_{x2} + G\Phi_{x3} \tag{2.4}
$$

where $K = K(x,\alpha)$ is the Cauchy function of the equation $L[y] = 0$, which can be defined as (cf Zoryj, 1982)

$$
K(x,\alpha) = \int\limits_{\alpha}^{x} \frac{1}{f(t)}(x-t)U(t,\alpha)\,dt \tag{2.5}
$$

The fundamental solution to Eq (2.1) is

$$
\Phi(x,\alpha) = K(x,\alpha)\Theta(x-\alpha) \tag{2.6}
$$

where $\Theta(x)$ denotes the Heaviside function.

The following notation has been introduced in Eqs (2.4) ÷ (2.6)

$$
\begin{aligned}
& K_{x1} = K(x,x_1) && \Phi_{xi} = \Phi(x,x_i) && i = 2,3 \\
& \dot{K}_{x1} = \frac{\partial K}{\partial \alpha}\Big|_{\alpha=x_1}
\end{aligned}
\tag{2.7}
$$

where

$K(x,\alpha)$ – influence function

C_i – arbitrary constants, $i = 0,1,2,3$.

It has been shown by Zoryj (1982) and (1987) that $U = U(x,\alpha)$ is the solution to the following problem

$$
\begin{aligned}
& U'' + \frac{1}{f(x)}PU = 0 \\
& U(\alpha) = 0 \qquad\qquad U'(\alpha) = 1
\end{aligned}
\tag{2.8}
$$

The function U can be written in the form of power series (cf Zoryj, 1982 and 1987)

$$U(x,\alpha) = \sum_{K=0}^{\infty} (-P)^K U_K(x,\alpha) \tag{2.9}$$

where

$$\begin{aligned} U_K(x,\alpha_K) &= \int\limits_{\alpha}^{x} \frac{x-t}{f(x)} U_K(t,\alpha)\, dt \\ U_0(x,\alpha) &= x - \alpha \qquad\qquad K = 1,2,... \end{aligned} \tag{2.10}$$

The series is convergent for any P and $x,\alpha \in [a,b]$ provided that: $\max\limits_{a \leq x \leq b} [1/f(x)] = M < \infty$.

3. Solution to the problem

From Eqs (2.4) ÷ (2.6) we have

$$fy" = C_2 U_{x1} + C_3 \dot{U}_{x1} - RU_{x2}\Theta_{x2} + GU_{x3}\Theta_{x3} \tag{3.1}$$

Substituting Eqs (2.4) and (2.8) into Eq (2.2), we have: $C_0 = 0$ and $C_3 = 0$. Employing some of the conditions (2.2) and (2.3) results in the following

$$\begin{aligned} & C_2 U_{b1} + R_3 U_{b2} = 0 \\ & C_1 P + C_2 (U'_{b1} + PK'_{b1}) - R(U'_{b2} + PK'_{b2}) = -G \\ & C_1 (x_2 - a) + C_2 K_{21} - R\frac{1}{c} = 0 \end{aligned} \tag{3.2}$$

From Eqs (3.2), we have

$$\begin{aligned} C_1 &= \frac{G}{\Delta(x_2 - a)} \Big(\frac{1}{c} U_{b1} - K_{21} U_{b2} \Big) \\ C_2 &= \frac{G}{\Delta} U_{b2} \qquad\qquad R = \frac{G}{\Delta} U_{b1} \end{aligned} \tag{3.3}$$

where

$$\Delta = U_{b1}U'_{b2} - U'_{b1}U_{b2} + P\Big[U_{b2}\Big(K'_{b2} - \frac{1}{c(x_2-a)}\Big) - U_{b2}\Big(K'_{b1} - \frac{K_{21}}{x_2-a}\Big)\Big] \tag{3.4}$$

By taking into consideration the following relation proposed by Zoryj (1987)

$$K(x,\alpha) \equiv \frac{1}{P}[(x-\alpha) - U(x,\alpha)] \tag{3.5}$$

Eq (3.4) can be transformed to a simpler form

$$\Delta = U_{ba} - U_{b2} + PD \tag{3.6}$$

where

$$D = \frac{1}{x_2-a}\Big(U_{b2}K_{2a} - \frac{1}{c}U_{ba}\Big) \tag{3.7}$$

The above yields the following solution to the problem defined by Eq (2.1) ÷ (2.3)

$$y(x) = \frac{G}{\Delta}[-(x-a)D + U_{b2}k_{xa} - U_{ba}\Phi_{x2}] + G\Phi_{x3} \tag{3.8}$$

which describes the beam deflection curve. Substituting $x = b$ one obtains the deflection of the beam end $y(b)$

$$y(b) = \frac{G}{\Delta}\Big\{-(b-a)D + \frac{1}{P}[(b-a)U_{b2} - (b-x_2)U_{ba}]\Big\} \tag{3.9}$$

It should be noted, that the lawest solution to the equation $\Delta(P) = 0$, where Δ is defined by Eqs (3.6) and (3.7) corresponds to the Euler critical force, that is $P = P_E$. As it has been expected for $P \to P_E$ the relations (3.8) and (3.9) lose their sense, as $y \to \infty$ is tensile force, than the sign " + " in Eqs (3.5) ÷ (3.9) should be replaced by the sign " − ", but of course in such a case $\Delta(P) \neq 0$.

4. The particular ceses

4.1. From the engineering point of view, it is a very interesting case, when $c \to \infty$. It means that the support at the point $x = x_2$ is absolutely rigid. Then in expressions (3.6), (3.7) and (3.8), (3.9)

$$D = \frac{1}{x_2-a}U_{b2}K_{2a} \tag{4.1}$$

It is obvious that, when $c = \infty$ and $x_2 \to b$ then $D \to 0$, $\Delta \to U_{ba}$, $y(b) \to 0$.

When $c = \infty$ and $x_2 \to a$, after solution for the indeterminacy of type $\frac{0}{0}$, we have found

$$y(b) \to -\frac{1}{U_{ba}P}[(b-a)\dot{U}_{ba} + U_{ba}] \tag{4.2}$$

At $P \to 0$, taking into consideration that $U_{ba} \to -1$ and applying the suitable formulas, it has been obtained

$$y(b) \to \frac{1}{P}[-(b-a)-P\dot{U}_1(b-a)-...+(b-a)-PU_1+...] \to [-(b-a)\dot{U}_1-U_1]\Big|_{\substack{x=b\\ \alpha=a}}$$

and finaly

$$y(b) \to \int\limits_a^b \frac{1}{f(s)}(b-s)^2\; ds \tag{4.3}$$

The relations (4.2) and (4.3) show a good agreement with the known solution (cf Zoryj, 1987).

4.2. In the peculiar case of the constant cross-section beam $f(x) = f_0 = EJ_0$, it has been obtained

$$U(x,\alpha) = \frac{1}{k}\sin k(x-\alpha) \qquad\qquad k^2 = \frac{P}{f_0} \tag{4.4}$$

$$K(x,\alpha) = \frac{1}{k^2 f_0}\Big[x - \alpha\frac{1}{k}\sin k(x-\alpha)\Big] \tag{4.5}$$

By replacing (4.4) and (4.5) into Eqs (3.8) and (3.9), it is possible to obtain the deflection curve and the deflection value at the end of a beam.

4.3. When the compressive force $P = 0$, we have

$$U(x,\alpha) = x - \alpha \qquad\qquad K(x,\alpha) = \int\limits_\alpha^x \frac{1}{f(s)}(x-s)(s-\alpha)\; ds \tag{4.6}$$

$$\Delta = x_2 - a \qquad\qquad D_0 = \frac{1}{x_2 - a}\Big[(b-x_2)K_{2a} - \frac{b-a}{c}\Big] \tag{4.7}$$

The deflection of a beam at its end $x = b$, which has been determined from Eq (3.9), has the form

$$y(b) = \frac{G}{x_2 - a}[-(b-a)(D_0 + K_{b2}) + (b-x_2)K_{ba}] \tag{4.8}$$

When we assume the constant cross-section $f(x) = f_0$ =const, i.e.

$$K(x,\alpha) = \frac{1}{6f_0}(x-\alpha)^3 \tag{4.9}$$

we have the deflection expression as follows

$$x(l) = G\Big[\frac{l^2}{cx_2^2} + \frac{1}{3f_0}l(l-x_2)^2\Big] \tag{4.10}$$

at $a = 0$ and $b = l$.

In the case of the rigid support of this beam $c \to \infty$, it has been obtained

$$y(l) \to \frac{1}{3f_0}Gl(l-x_2)^2 \tag{4.11}$$

At $x_2 \to 0$ and $x_2 \to l$ it has been obtained

$$y(l) = \frac{Gl^3}{3f_0} \qquad\qquad y(l) = 0 \tag{4.12}$$

what is comparable to the formulas (4.1) and (4.3).

The first form shows, that two pivot supports, which are close together enough, are equivalent to the rigid fixing.

This conclusion results also from the formula (4.3), that is adequate to the variable cross-section of a semi-beam.

4.4. The variable rigidity of the tapered beam is taken into consideration with help of the following expression (cf Timoshenko, 1971)

$$f(x) = EJ_0(1-\gamma x)^4 \tag{4.13}$$

where γ – convergence factor.

In this case the analogical formulas have the form

$$U(x,\alpha) = \frac{1}{\varphi(p,x,\alpha)}\sin[(x-\alpha)\varphi(p,x,\alpha)] \tag{4.14}$$

$$\varphi(p,x,\alpha) = \frac{\sqrt{p}}{(1-\gamma x)(1-\gamma\alpha)} \tag{4.15}$$

$$p = \frac{Pl^2}{EJ_0} \tag{4.16}$$

5. Example of calculations

The stability of the cantilever beam under conservative force has been considered as the example of calculations. The exponential change in the rigidity of a beam has been assumed

$$f(x) = EJ_0^{-\nu\xi} \tag{5.1}$$

where

$$\xi = \frac{x}{l} \qquad -\infty < \nu < +\infty$$

and p – parameter of the load (4.15).

Upper and lower estimators of critical Euler load p_- and p_+ calculated from the known Bernshtein formulas (cf Bernshtein and Keropian, 1960)

$$p_- = \frac{1}{\sqrt[4]{b_4}} \qquad p_+ = \sqrt{\frac{2}{b_2\Big[1 + 2b_4\Big(\frac{1}{b_2^2} - 1\Big)\Big]}} \tag{5.2}$$

has been obtained on the basis of the formula (3.6).

The results of calculations of the upper and lower estimators of the critical Euler load are presented in the Table.

Table

ν	0	1	−1	2	−2	10	−10	−100
p_-	2.45	1.76	3.26	1.19	4.07	0.0064	11.66	101.53
p_+	2.45	1.86	3.30	1.29	4.13	0.0069	11.68	101.53

6. Conclusions

- The presented method gives possibilities of the solution to the problem of deflection and stability of a variable cross-section beam in a closed analytic form (formulas (3.2) $\div$ (3.6). It is obvious, that these formulas holds also for the optional integrable function $1/f(x)$, what gives a lot of possibilities of using this method.

- It is possible to observe accuracy of calculations on the example of critical Euler force compressing a cantilever beam (table) by using the simplest Bernshtein estimators.

- A lot of problems in engineering, which appear in structural design of variable cross-section, have been analysed in this model of a beam.

References

1. BERNSHTEIN C.A., KEROPYAN K.K, 1960, *Opredelenie zastot kolebaniĭ sterzhnevykh sistem metodom spektralnoĭ funkcii,* Gosstroĭizdat, Moskva
2. JAROSZEWICZ J., ZORYJ L., 1983, *O zastosowaniu metody szeregów charakterystycznych do analizy giętnych drgań własnych pręta z uwzględnieniem masy własnej,* Zeszyty Naukowe PB, Mechanika, 1, Białystok
3. JAROSZEWICZ J., ZORYJ L., 1985, *Drgania giętne belki wspornikowej o zmiennym przekroku,* Rozprawy Inżynierskie, **33**, 4, 537-547
4. TIMOSHENKO S.P., 1971, *Ustoĭchivost' sterzhneĭ, plastin, obolochek,* Nauka, Moskva
5. TIMOSHENKO S.P., GUDER G., 1979, *Teoriya uprugosti,* Nauka, Moskwa
6. ZORYJ L.M., 1982, *Ob universalnykh kharakteristicheskikh uravneniyakh v zadachakh kolebanii i ustoĭchivosti uprugikh sistem,* Mekhanika Tvërdogo Tela, 6, 155-162
7. ZORYJ L.M., (EDIT.), 1987, *Raschëty i ispytaniya na prochnost,* Metodicheskiya rekomendacii MR 213-87, GODDTANDART SSSR, Moskva

Analiza ugięcia i stateczności belki o zmiennym przekroju metodą funkcji wpływu

Streszczenie

W pracy zastosowano metodę funkcji wplywu do rozwiązania problemu wyznaczenia ugięcia i obciążenia krytycznego belki o zmiennym przekroju. Otrzymano funkcję opisującą linię ugięcia sprężyście podpartej belki o zmiennej sztywności giętnej przy pomocy funkcji Cauchy'ego w postaci szeregów potęgowych. Otrzymano ogólną postać równania charakterystycznego, na podstawie którego możliwe jest obliczenie estymatów krytycznego obciążenia w sensie Eulera. Rozpatrzono w szczególach kilka przykładów belek, często spotykanych w praktyce inżynierskiej. Porównanie otrzymanych wyników z dobrze znanymi wynikami pokazalo wysoką dokladność zaproponowanej metody.

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