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ACCURACY ANALYSIS OF STATISTICAL LINEARIZATION METHODS IN NONLINEAR CONTINUOUS SYSTEMS DESCRIBED BY RANDOM INTEGRAL EQUATIONS

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The aim of this paper is to present the accuracy analysis of dynamic systems described by nonlinear Volterra integral equations of the second kind. A nonstationary dynamic system is considered as well as a stationary one. The technique of harmonic analysis of noncommutative causal operators from $L^1(\mathcal{R}; L(\mathcal{R}^n, \mathcal{R}^n))$ is used to analyse the special Banach algebras of integral operators being limiting filters. A bound of the statistical linearization error between the exact and approximate solutions is given. The conditions under which mean-square continuous exact solutions exist are considered too. To compare presented methods with exact analytical solutions for different statistical linearization methods an illustrative example is given.

1. Introduction

The method of nonlinear stochastic systems linearization has been used widely in structural dynamics analysis (cf e.g. Atalik and Utku (1976); Booton (1954); Bunke (1972); Caughey (1963); Gutowski and Świetlicki (1986); Kazakov (1975); Krasovskiĭ (1974); Piszczek (1970), (1972) and (1982); Piszczek and Nizioł (1986); Pugachev (1962); Roberts and Spanos (1990); Skrzypczyk (1994a,b); Sobczyk (1973); Spanos and Iwan (1978); Spanos (1981)). The rich bibliography is given in monographs (cf Gutowski and Świetlicki (1986); Kazakov (1975); Piszczek and Nizioł (1986); Roberts and Spanos (1990); Sobczyk (1973)). There are several different methods of nonlinear stochastic problems linearization. The best known is, probably, the statistical linearization technique invented independly by Booton (1954) and Kazakov (1975). In principle,

the statistical linearization should be applied in the case of weak nonlinearities, but it turns out that this method gives also good results in the case of severe nonlinearities, even when discontinuous terms appear. As pointed out above, the related technique was very widely used,but the exact analytical accuracy analysis was omitted. Known cases, in which linearization errors were analysed can be classified into three groups.

The first one, contains papers in which approximate linearized solutions are compared with exact ones, it works for analytically solvable, usually very simple, dynamical systems driven by white noise or stationary type processes (cf Gutowski and Świetlicki (1986); Kazakov (1975); Krasovskiĭ (1974); Piszczek (1970), (1972) and (1982); Piszczek and Niziol (1986); Skrzypczyk (1994a)).

In the second one, generally,the accuracy of linearization methods is verified against the Monte-Carlo simulation for different dynamical systems, and for a variety of external excitations (cf e.g. Atalik and Utku (1976); Roberts and Spanos (1990); Sobczyk (1973); Spanos and Iwan (1978)).

The third group contains attempts to obtain some theoretical results which are not satisfactory enough because of some aspects of mathematical exactness (cf Artem'ev and Stepanov (1978); Holtzman (1968); Kazakov (1975); Krasovskiĭ (1974)). Our approach is a continuation of the papers (cf Skrzypczyk (1985) and (1986)), however we dwell more on some practical aspects of the mathematical analysis.

The aim of the present paper is to illustrate the exact accuracy analysis of statistical linearization technique, applied to multidegree-of-freedom stochastic systems, based on the modern system theory and the theory of stochastic processes with values in Banach spaces. We adapt the technique of stochastic stability to obtain a bound of the error between the exact and the approximate linearized solutions, respectively.

2. Definitions and notion

The following symbols are in use: $\mathcal{R}^n$ was reserved for the set of n-dimensional reals, $\mathcal{R}^1 = \mathcal{R}$, $\mathcal{C}^n$ – the set of n-dimensional complex numbers, $\mathcal{R}_+^n$ – the set of nonnegative n-dimensional reals.

$A^\top$ and $x^\top$ denote, respectively, the transpose of the matrix A and the vector x. A^* and x^* denote the complex conjugate and transpose of the matrix A and the vector x, respectively, $\bar{A}$ and $\bar{x}$ the complex conjugate of the matrix A and the vector x, respectively (i.e. $A^* = \bar{A}^\top$). The Euclidean

norm of the vector x is denoted by $|x|$, the scalar product of vectors x and y by $<x,y>$ and the determinant of the matrix A by $\det(A)$, thus we have

$$|x| = (x^*x) \qquad\qquad <x,y> = x^*y$$

For any given matrix A, the square roots of the eigenvalues of A^*A are called the singular values of A. For any matrix A, we use the notation $\sigma_{\max}(A)$ to denote the largest singular value of A and $\sigma_{\min}(A)$ to denote the smallest singular value of A. Singular values are always nonnegative real numbers since A^*A is always the Hermitian (positively semidefinite). In analogous way we denote by $\lambda_{\max}(A)$ $(\lambda_{\min}(A))$ the largest eigenvalue (the least one) of the Hermitian matrix $(A + A^*)/2$.

An ordered triplet $(\Omega, \mathcal{B}, P)$ will be called a probability space (cf Gihman and Skorohod (1980)). Let $x(\omega)$, $\omega \in \Omega$ be a measurable mapping $\Omega \to \mathcal{C}^n$ (event. $\mathcal{R}^n$), it will be called a complex (real) random value. Further we use the notion

$$E\{x(\cdot)\} := \int_\Omega x(\omega)P(d\omega)$$

$$\|x(\cdot)\|_p := \Big(\int_\Omega |x(\omega)|^p P(d\omega)\Big)^{1/p} \qquad 1 \le p < \infty$$

Let $x(t,\omega)$, $t \in \mathcal{R}^1$, $\omega \in \Omega$, be a second-order, mean-square continuous, stationary in wide sense, stochastic process with values in $\mathcal{R}^n$ and $E\{x(t,\omega)\} = 0$. It has a spectral representation

$$x(t,\omega) = \int_{\mathcal{R}} \exp(\mathrm{i}ut)\zeta(du)$$

where $\zeta(\cdot)$ is the spectral process of $x(\cdot)$, and

$$E\{\zeta(du)\zeta^*(du)\} = F^x(du) \qquad\qquad \mathrm{tr}F^x(\mathcal{R}) < \infty$$

Here the nonnegatively definite matrix $F^x(\cdot)$ is the structural matrix function of the process $x(\cdot)$ (cf Gihman and Skorohod (1980); Rozanov (1974)). The results presented below are only in part based on the monograph by Gihman and Skorohod (1980).

Let $L^2(x)$ be the closed linear span in $L^2(\Omega, \mathcal{B}, P)$ of components of $x(t,\omega)$, $t \in \mathcal{R}^1_+$, with scalars from the field $\mathcal{C}^1$. We define a

space $L^2(F^x; L(\mathcal{R}^n, \mathcal{R}^n))$ of $n \times n$-dimensional measurable matrix functions defined on $\mathcal{R}_+^1$ with the following property: the matrix function $k(\cdot) \in L^2(F^x; L(\mathcal{R}^n, \mathcal{R}^n))$ iff $\int_{\mathcal{R}} \mathrm{tr}\left\{k(iu)F^x(du)k^*(iu)\right\} du < \infty$.

We deal quite freely with spaces of the form $C(\mathcal{R}; L^2(\Omega, \mathcal{B}, P))$, $L^{2,2}(\mathcal{R}) = L^2(\mathcal{R}; L^2(\Omega, \mathcal{B}, P))$, $L^1(\mathcal{R}; L^2(\Omega, \mathcal{B}, P))$, etc., where for example, $C(\mathcal{R}; L^2(\Omega, \mathcal{B}, P))$ is the space of continuous maps of into the Hilbert space of second-order random values and the notion is generally similar to that used in the theory of generalized stochastic processes with values in Banach spaces and mixed-norm spaces. The inner products in the spaces L^2 and $L^{2,2}$ are denoted by $<\cdot,\cdot>_2$ and $<\cdot,\cdot>_{2,2}$ respectively.

3. Statistical linearization of nonstationary systems

Consider the feedback nonlinear dynamic system described by the equation

$$x = KFx + z \tag{3.1}$$

where

- z – n-dimensional stochastic process defined on $\mathcal{R}_+^1 \subset \mathcal{R}$
- F – nonlinear operator
- K – causal linear integral operator.

Assume, that the operators F and K, respectively, have the following forms

$$(Fx)(t,\omega) := f(t, x(t,\omega)) \qquad t \in \mathcal{R}_+^1 \tag{3.2}$$

where $f : \mathcal{R}_+^1 \times \mathcal{R}^n \to \mathcal{R}^n$ is a nonlinear function and

$$(Kx)(t,\omega) := \int_0^t k(t,s)x(s,\omega)\, ds \qquad t \in \mathcal{R}_+^1 \tag{3.3}$$

where the kernel $k(t,s)$ is defined on Δ and $\Delta := \left\{(t,s) : t,s \in \mathcal{R}_+^1, 0 \le s \le t < \infty\right\}$, $k(t,s) = 0$ for $s > t$, $k(t,\cdot) \in L^1_{loc}(\mathcal{R}_+^1)$ and integration is over the set $[0,t[\subset \mathcal{R}_+^1$.

Generally the method of statistical linearization replaces the nonlinear problem desribed by a nonlinear integral equation of the form

$$x(t,\omega) = z(t,\omega) + \int_0^t k(t,s)f(s,x(s,\omega))\, ds \qquad t \in \mathcal{R}_+^1 \tag{3.4}$$

by a linear problem of the following form

$$y(t,\omega) = z(t,\omega) + \int_0^t k(t,s)l(s,y(s,\omega))\,ds \qquad t \in \mathcal{R}_+^1 \tag{3.5}$$

where $l(t,y) := C_n(t)y + c(t)$, $y = [y_1, y_2]^\top \in \mathcal{R}^n$, $y_1 \in \mathcal{R}^p$, $p \leq n$. We assume further that

$$C_n y = [C, 0]y = Cy_1 \qquad C : \mathcal{R}^p \to \mathcal{R}^n$$

Thus, the formulation of statistical linearization method (SLM) is similar to that given by Kazakov (1975), Booton (1954) in the proposed analysis assumed that $c(\cdot) \equiv 0$.

Naturally, the linearized Eq (5) will be a good approximation of the nonlinear problem (1 = 4) if the exact solution $x(\cdot,\cdot)$ and the approximate one $y(\cdot,\cdot)$ differ a "little" from each other. The accuracy of SLM, i.e. a measure of the difference $|x(t,\omega) - y(t,\omega)|$, for $t \in \mathcal{R}_+^1$, can be defined in many ways, e.g.

$$\varepsilon = \sup_{t\in[0,\infty[} \sqrt{E\left\{|x(t,\omega) - y(t,\omega)|^2\right\}} \tag{3.6}$$

or

$$\tilde{\varepsilon} = \sqrt{\int_0^\infty E\left\{|x(t,\omega) - y(t,\omega)|^2\right\} dt} \tag{3.7}$$

In such a formulation of a statistical linearization problem a fundamental fault appears. The point is, that the exact analytical solution of such formulated problem is not known. As a test of SLM accuracy we take usually other forms of quality functionals, different from Eqs (3.6) and (3.7). There is no one mind in this problem (cf Caughey (1963); Kazakov (1975)).

Assume that the linearization coefficients C_n and c are chosen on the ground of a certain functional criterion.

We will discuss now problems of SLM accuracy in the nonstationary case. We first omit the existence and uniqueness of solutions.

- LEMMA 1. Assume that the following conditions are satisfied:

  (i) there exist real numbers $\lambda \in \mathcal{R}^1$, $r \in \mathcal{R}_+^1$ such that, for each $t \in \mathcal{R}_+^1$, and some random functions $x_1, x_2 \in C(\mathcal{R}_+^1; L^2(\Omega, \mathcal{B}, P))$

the following relation is true

$$E\Big\{|f(t,x_1(t,\omega)) - f(t,x_2(t,\omega)) - \lambda[x_1(t,\omega) - x_2(t,\omega)]|^2\Big\} \leq$$

$$\leq r^2 E\Big\{|x_1(t,\omega) - x_2(t,\omega)|^2\Big\}$$

(ii) the resolvent kernel associated with $k(t,s)$, say $r(t,s)$, exists on Δ and is such that

$$\|R\|_c \leq |\frac{\lambda}{r}|$$

where

$$\|R\|_c := \sup_{t\in[0,\infty[} \int_0^t |r(t,s)|\, ds$$

Then there is a real constant $a > 0$, such that all exact and approximate solutions (if exist) of equations of the form (3.4) satisfy the inequality

$$\|x(t,\omega) - y(t,\omega)\|_{C(\mathcal{R}_+^1;L^2(\Omega,\mathcal{B},P))} \leq a\rho(l)$$

where

$$\rho(l) := \sup_{t\in[0,\infty[} \sqrt{E\Big\{|f(t,y(t,\omega)) - C_n(t)y(t,\omega) - c(t)|^2\Big\}}$$

defines the error of statistical linearization.

Proof. Assume, that the solutions of Eqs (3.1)=(3.4) and (3.5) exist and are measurable stochastic processes. We can rewrite Eqs (3.4) and (3.5) in the equivalent forms

$$x - \lambda Kx = KFx - \lambda Kx + z \tag{3.8}$$

$$y - \lambda Ky = K(C_n y + c - \lambda y) + z \tag{3.9}$$

It follows from Eq (3.8) and (3.9) that

$$\begin{aligned}(I - \lambda K)(x - y) &= K(Fx - \lambda x - C_n y - c + \lambda y) = \\ &= K(Fx - Fy - \lambda(x - y)) + K(Fy - C_n y - c)\end{aligned} \tag{3.10}$$

Let recall first that the resolwent kernel $r(t,s)$ satisfies the equation

$$r(t,s) = k(t,s) + \int_s^t r(t,\tau)k(\tau,s)\, d\tau \tag{3.11}$$

We refer the reader to Skrzypczyk (1987) for details. From Eqs (3.10) and (3.11), we obtain

$$x(t,\omega) = y(t,\omega) +$$

$$+\frac{1}{\lambda}\int_0^t r(t,s)\Big(f(s,x(s,\omega)) - f(s,y(s,\omega)) - \lambda[x(s,\omega) - y(s,\omega)]\Big)\,ds +$$

$$+\frac{1}{\lambda}\int_0^t r(t,s)\Big(f(s,y(s,\omega)) - C_n(s)y(s,\omega) - c(s)\Big)\,ds$$

It is easy to check that

$$\sup_{t\in[0,\infty[}\|x(t,\omega) - y(t,\omega)\|_2 \leq$$

$$\leq \frac{1}{|\lambda|}\|R\|_c r \sup_{t\in[0,\infty[}\|x(t,\omega) - y(t,\omega)\|_2 + \frac{1}{|\lambda|}\|R\|_c\rho(l) \tag{3.12}$$

It follows directly from Eq (3.12) and the Assumption (ii) that the statement of Lemma 1 is true.

Remark. The condition (ii) is a restrictive one. To satisfy it, sometimes, it is necessary, but often possible, to take into considerations some special weight function spaces of continuous functions.

When the nonlinearity is a time-invariant function, and $k(t,s) = k(t-s)$ for all $(t,s) \in \Delta$, we can formulate more precise results, similar to that of Lemma 1.

- LEMMA 2. Assume that the following three conditions are satisfied:

  (i) there exist real numbers $\lambda \in \mathcal{R}^1$, $r \in \mathcal{R}^1_+$ such that, for each $t \in \mathcal{R}^1_+$, and some random functions $x_1, x_2 \in L^2(\mathcal{R}^1_+; L^2(\Omega,\mathcal{B},P))$ the following relation is true

$$\int_0^t E\Big\{|f(t,x_1(t,\omega)) - f(t,x_2(t,\omega)) - \lambda[x_1(t,\omega) - x_2(t,\omega)]|^2\Big\}\,dt \leq$$

$$\leq r^2\int_0^t E\Big\{|x_1(,\omega) - x_2(t,\omega)|^2\Big\}\,dt$$

(ii) $\sup_{u\in\mathcal{R}}\sigma_{\max}\left(\widehat{d}(iu)\right) < \frac{1}{r}$ where $\widehat{d}(iu) = [I - \lambda\widehat{k}(iu)]^{-1}\widehat{k}(iu)$

(iii) $\det[I - \lambda\widehat{k}(iu)] \neq 0 \quad \forall u \in \mathcal{R}$

Then there is a real constant $a > 0$, such that all exact and approximate solutions (if exist) of dynamic systems (3.4) satisfy the inequality

$$\|x(t,\omega) - y(t,\omega)\|_{L^2(\mathcal{R}^1_+;L^2(\Omega,\mathcal{B},P))} \leq \widetilde{a}\rho(l)$$

where

$$\widetilde{\rho}(l) := \sqrt{\int_0^t E\left\{|f(t,y(t,\omega)) - C_n(t)y(t,\omega) - c(t)|^2\right\} dt}$$

defines the error of statistical linearization.

Proof. It is quite similar to that of Lemma 1.

Remark. In the Lemma 2 the constant λ can be replaced by some stationary and causal linear operator of the class $B_P(W_P)$ (cf Skrzypczyk (1987)). In this case the condition (iii) takes the form

(iii)' $\det[I - \widehat{k}(iu)\widehat{b}(iu)] \neq 0 \quad \forall u \in \mathcal{R}$
where $b(\cdot)$ is the operator kernel, and
$\widehat{d}(iu) = [I - \widehat{k}(iu)\widehat{b}(iu)]^{-1}\widehat{k}(iu)$

The proof is almost the same.

- LEMMA 3. Assume, that the following conditions are satisfied:

(i) $\sup_{u\in\mathcal{R}}\sigma_{\max}\left(\widehat{k}(iu)\right) < \infty$

(ii) there is a real number $\alpha \in \mathcal{R}^1$ such that, for each $t \in \mathcal{R}^1_+$, and every stochastic processes $x_1, ; x_2 \in L^2(\mathcal{R}^1_+; L^2(\Omega, \mathcal{B}, P))$ the following relation holds

$$\mathrm{Re}\int_0^t E\left\{< f(x_1(t,\omega)) - f(x_2(t,\omega)), x_1(t,\omega) - x_2(t,\omega) >\right\} dt \geq$$

$$\geq \alpha \int_0^t E\left\{|f(x_1(t,\omega)) - f(x_2(t,\omega))|^2\right\} dt$$

(iii) $\sup_{u\in\mathcal{R}} \lambda_{\max}\left(\widehat{k}(iu)\right) < \alpha$

Then the statement of Lemma 2 remains valid.

Proof. Assume, that there are, not necessary unique, solutions of Eqs (3.4) and (3.5). Subtracting these equations, we get

$$x - y = K(Fx - C_n y - c) = K(Fx - Fy) + K(Fy - C_n y - c) \quad (3.13)$$

Multiplying Eq (3.13) scalarly by $Fx - Fy$, we obtain

$$\begin{aligned} <Fx - Fy, x - y>_{2,2} &= <Fx - Fy, K(Fx - Fy)>_{2,2} + \\ &+ <Fx - Fy, K(Fy - C_n y - c)>_{2,2} \end{aligned} \quad (3.14)$$

Following the Assumption (iii) we get

$$\begin{aligned} \alpha\|Fx - Fy\|_{2,2}^2 &\leq (\sup_2 K)\|Fx - Fy\|_{2,2}^2 + \\ &+ \|K\|_2\|Fx - Fy\|_{2,2}\|Fy - C_n y - c\|_{2,2} \end{aligned} \quad (3.15)$$

It is natural to assume $\|Fx - Fy\|_{2,2} \neq 0$, and it follows from (i), (iii) and inequality (3.15), that

$$\begin{aligned} &(\alpha - \sup_2 K)\|Fx - Fy\|_{2,2} \leq \|K\|_2\|Fy - C_n y - c\|_{2,2} \\ &\|Fx - Fy\|_{2,2} \leq \frac{\|K\|_2}{\alpha - \sup_2 K}\tilde{\rho}(l) \end{aligned} \quad (3.16)$$

With inequality (3.16), from Eq (3.13) it follows that

$$\|x - y\|_{2,2} \leq \|K\|_2\|Fx - Fy\|_{2,2} + \|K\|_2\tilde{\rho}(l) \leq \left(\frac{\|K\|_2^2}{\alpha - \sup_2 K} + \|K\|_2\right)\tilde{\rho}(l)$$

This statement ends the proof.

- LEMMA 4. Assume, that the Condition (i) of Lemma 3 is satisfied and additionally:

(i) there is a real number $\beta \in \mathcal{R}^1$ such that for every $t \in \mathcal{R}_+^1$, and every stochastic process $x_1, x_2 \in L^2(\mathcal{R}_+^1; L^2(\Omega, \mathcal{B}, P))$ the

following relation holds

$$\operatorname{Re} \int_0^t E\Big\{ < f(x_1(t,\omega)) - f(x_2(t,\omega)), x_1(t,\omega) - x_2(t,\omega) > \Big\}\, dt \leq$$

$$\leq \beta \int_0^t E\Big\{ |f(x_1(t,\omega)) - f(x_2(t,\omega))|^2 \Big\}\, dt$$

(ii) $\inf\limits_{u \in \mathcal{R}} \lambda_{\min}\Big(\widehat{k}(iu)\Big) > \beta$

Then the statement of Lemma 3 remains true.

Proof. It is similar to that of Lemma 3.

The results of Lemmas $1 \div 4$ illustrate the fact, that the "distance" between the exact solution of Eq (3.4) and the approximate one of Eq (3.5), remains in a strict connection with the linearization error introduced in Eq (3.6) or (3.7).

- THEOREM 5. Assume, that the following four assumptions are true.

(i) There exist a constant $r > 0$ and a matrix B, satisfying

$$\|Fx_1 - Fx_2 - B(x_1 - x_2)\|_{2,2} \leq r\|x_1 - x_2\|_{2,2}$$

for every $x_1, x_2 \in L^2(\mathcal{R}_+^1; L^2(\Omega, \mathcal{B}, P))$

(ii) $k(\cdot) \in L^1(\mathcal{R}_+^1)$ and $\inf\limits_{u \in \mathcal{R}} |\det[I - \widehat{k}(iu)B]| > 0$

(iii) $\sup\limits_{u \in \mathcal{R}} r\sigma_{\max}\Big(\widehat{d}(iu)\Big) < 1$

where $\widehat{d}(\cdot)$ is defined in (iii)' of Lemma 2

(iv) There is a resolvent for the linearized integral kernel $k(t-s)C_n(s)$ in the space $L^2(\mathcal{R}_+^1)$.

Then there exists a unique exact solution to the nonlinear Eq (3.4). The exact solution can be obtained by the method of successive approximations, and the linearization error remains the same as in Lemmas $2 \div 4$.

Proof. Let the solution $\tilde{y}$, to the linearized Eq (3.5), be the first approximation for the method of successive approximations, applied to Eq (3.4).

On the assumption (ii) we can write

$$x = (I - KB)^{-1} K \tilde{F} x + (I - KB)^{-1} z \tag{3.17}$$

where $\tilde{F}x = Fx - Bx$. Now define the operator A as follows

$$Ax := (I - KB)^{-1} K \tilde{F} x + (I - KB)^{-1} z$$

We note, that Eq (3.5)=(3.17) is equivalent to the form

$$x = Ax \tag{3.18}$$

As mentioned above, we will investigate the convergence of the method of successive approximations, applied to Eq (3.18). Let

$$\begin{aligned} x_0 &= \tilde{y} \\ x_{i+1} &= Ax_i \qquad\qquad i = 0, 1, 2, \ldots \end{aligned} \tag{3.19}$$

with x_0 being the statistical linearization approximation. The approximation x_0 is the solution to Eq (3.5). Following (iv) we can write

$$x_0 = (I - KC_n)^{-1}(Kc + z) \tag{3.20}$$

and

$$\begin{aligned} x_1 - x_0 &= (I - KB)^{-1} K \tilde{F} x_0 + (I - KB)^{-1} z - x_0 = \\ &= (I - KB)^{-1} KFx_0 - (I - KB)^{-1} KBx_0 + (I - KB)^{-1} z - x_0 \end{aligned} \tag{3.21}$$

Then

$$x_0 - KBx_0 = KLx_0 - KBx_0 + z$$

and

$$x_0 = (I - KB)^{-1} K(Lx_0 - Bx_0) + (I - KB)^{-1} z \tag{3.22}$$

Now with Eqs (3.21) and (3.22) we have

$$x_1 - x_0 = (I - KB)^{-1} K(Fx_0 - Lx_0) \tag{3.23}$$

$$\|x_1 - x_0\|_{2,2} \leq \|(I - KB)^{-1} K\|_2 \tilde{\rho}(l) \tag{3.24}$$

We can estimate a difference between succeding iterations x_{i+1} and x_i with Eq (3.17)

$$x_{i+1} - x_i = (I - KB)^{-1} K(\tilde{F}x_i - \tilde{F}x_{i-1})$$

and next

$$\begin{aligned} \|x_{i+1} - x_i\|_{2,2} &\leq \|(I - KB)^{-1}K\|_2 \|Fx_i - Fx_{i-1}\|_{2,2} \leq \\ &\leq \|(I - KB)^{-1}K\|_2 r \|x_i - x_{i-1}\|_{2,2} \end{aligned} \tag{3.25}$$

Thus using the Assumption (iii), we have a Cauchy sequence and there is a limit $\tilde{x}$ of the sequence $\{x_i\}$, which is the exact solution to Eq (3.4).

Furthermore, with relations (3.24) and (3.25) we can estimate the linearization error $\|\tilde{x} - x_0\|_{2,2}$, as a function of $\tilde{\rho}(l)$. For details of this familiar argument see the proof of the contraction mapping fixed point theorem.

The last statement ends the considerations.

4. Statistical linearization of stationary systems

The statistical linearization method is usually used for nonlinear stationary dynamic systems. Consider the feedback nonlinear stationary system described by the equation

$$x = KFx + z \tag{4.1}$$

where

- z – n-dimensional, stationary stochastic process, defined on $\mathcal{R}$
- F – nonlinear, time-invariant operator
- K – limiting filter (cf Gihman and Skorohod (1980)).

Assume, that

$$(Fx)(t,\omega) = f(x(t,\omega)) \tag{4.2}$$

where $t \in \mathcal{R}$, $f : \mathcal{R}^n \to \mathcal{R}^n$, and the operator K is defined as follows

$$(K\eta)(t,\omega) := \int_{\mathcal{R}} \exp(iut)\hat{k}(iu)\zeta(du) \qquad t \in \mathcal{R} \tag{4.3}$$

and

$$\eta(t,\omega) = \int_{\mathcal{R}} \exp(iut)\zeta(du) \qquad t \in \mathcal{R}$$

The rest assumptions are the same as in Section 3.

Generally, similarly as for nonstationary systems, the SLM replaces Eq (4.1) by

$$y = KLy + z \tag{4.4}$$

where L is a linear, time-invariant operator, defined similarly as in the nonstationary case

$$Ly := C_n y + c$$

where C_n and c are time-invariant and notions are the same as in Section 3.

Assume, that the linearization coefficients C_n and c are chosen by some arbitrary method.

- THEOREM 6. Assume, that $z(t,\omega)$ is a stationary in strong-sense, mean-square continuous, second-order stochastic process and the following conditions are satisfied:

  (i) there is a limiting filter B, generated by a kernel $\hat{b}(iu)$ and

  $$\inf_{u\in\mathcal{R}} |\det[I - \hat{k}(iu)\hat{b}(iu)]| > 0$$

  (ii) $\inf_{u\in\mathcal{R}} |\det[I - \hat{k}(iu)C_n]| > 0$

  (iii) there is a constant $r > 0$, such that

  $$\sup_{t\in\mathcal{R}} E\left\{|f(x_1(t,\omega)) - f(x_2(t,\omega)) - B[x_1(t,\omega) - x_2(t,\omega)]|^2\right\} \leq$$
  $$\leq r^2 \sup_{t\in\mathcal{R}} E\left\{|x_1(t,\omega) - x_2(t,\omega)|^2\right\}$$

  for every $x_1, x_2 \in C(\mathcal{R}; L^2(\Omega, \mathcal{B}, P))$

  (iv) $\sup_{u\in\mathcal{R}} r\sigma_{\max}\left(\hat{a}(iu)\right) < 1$

  where $\hat{a}(iu) = [I - \hat{k}(iu)\hat{b}(iu)]^{-1}\hat{k}(iu)$.

  Then there is a stationary mean-square continuous solution to Eq (4.1). The error of SLM can be estimated as follows

  $$\sup_{t\in\mathcal{R}} \sqrt{E\left\{|\tilde{x}(t,\omega) - \tilde{y}(t,\omega)|^2\right\}} \leq \text{const} \sup_{t\in\mathcal{R}} \rho(l)$$

  where $\rho(l) = \sqrt{E\left\{|f(y(t,\omega) - C_n y(t,\omega) - c|^2\right\}}$ and $\tilde{y}(t,\omega)$ denotes a solution to the approximate Eq (4.4).

Proof. We first note that all operations are exactly parallel those of Theorem 5. Eq (4.1) is equivalent to

$$x - KBx = KFx - KBx + z = K\tilde{F}x + z \tag{4.5}$$

where $\tilde{F}x = Fx - Bx$. Let $\tilde{y}(t,\omega)$ be an approximate solution to Eq (4.1). Approximating Eq (4.2) is equvalent to

$$(I - KC_n)y = KC + z \tag{4.6}$$

On Assumption (ii) there is a stationary, mean-square continuous solution to the approximate Eq (4.4)=(4.6) and

$$y = (I - KC_n)^{-1}(Kc + z) \tag{4.7}$$

Let the limiting filter D be defined in the form

$$Dx := (I - KB)^{-1}K\tilde{F}x + (I - KB)^{-1}z$$

Assume, that $x_0 = \tilde{y}$ and

$$x_{i+1} = Dx_i \qquad \text{for} \quad i = 0, 1, 2, \ldots$$

define successive approximations of the solving procedure.

Following known, from Theorem 5, considerations we have

$$\begin{aligned} x_1 - x_0 = (I - KB)^{-1}KFx_0 - (I - KB)^{-1}KBx_0 + \\ +(I - KB)^{-1}z - x_0 \end{aligned} \tag{4.8}$$

$$x_0 = (I - KB)^{-1}K(Lx_0 - Bx_0) + (I - KB)^{-1}z \tag{4.9}$$

and with Eqs (4.8) and (4.9) we get

$$x_1 - x_0 = (I - KB)^{-1}K(Fx_0 - Lx_0)$$

The process $w_0 = x_1 - x_0$ is stationary, second-order, mean-square continuous, since it is the result of combination of stationary operations. With the stationarity, second-order and continuity properties assured, we can use the spectral representation

$$w_0(t,\omega) = \int_{\mathcal{R}} \exp(iut)\zeta_0(du)$$

where

$$E\left\{\zeta_0(du)\zeta_0^*(du)\right\} = F_0(du) \qquad \operatorname{tr}(F_0(\mathcal{R})) < \infty$$

Notice additionally, that

$$\sup_{t\in\mathcal{R}} E\left\{|w_0(t,\omega)|^2\right\} = \operatorname{tr}(F_0(\mathcal{R})) \leq \alpha^2 \sup_{t\in\mathcal{R}} \rho^2(l) \tag{4.10}$$

where

$$\alpha = \|(I - KB)^{-1}\|_{C(\mathcal{R};L^2(\Omega,\mathcal{B},P))}$$

$$\rho(l) = \sqrt{E\left\{|Fx_0 - Lx_0|^2\right\}}$$

We will estimate the difference

$$x_{i+1} - x_i = (I - KB)^{-1}K(\tilde{F}x_i - \tilde{F}x_{i-1}) \qquad i = 1, 2, 3, \ldots$$

All of the $w_i = x_{i+1} - x_i$ are stationary, second-order and continuous in the mean, following similar arguments, as in the case of w_0. We use the spectral representation

$$w_i(t,\omega) = \int_{\mathcal{R}} \exp(iut)\zeta_i(du) \tag{4.11}$$

where

$$E\left\{\zeta_i(du)\zeta_i^*(du)\right\} = F_i(du)$$

$$\operatorname{tr}(F_i(\mathcal{R})) < \alpha^2 E\left\{|\tilde{F}x_i - \tilde{F}x_{i-1}|^2\right\}$$

With Assumption (iii), we can write

$$\begin{aligned} \sup_{t\in\mathcal{R}} E\left\{|x_{i+1} - x_i|^2\right\} &= \operatorname{tr}(F_i(\mathcal{R})) \leq \alpha^2 r^2 \sup_{t\in\mathcal{R}} E\left\{|x_i - x_{i-1}|^2\right\} = \\ &= \alpha^2 r^2 \operatorname{tr}(F_{i-1}(\mathcal{R})) \end{aligned} \tag{4.12}$$

for all $i = 1, 2, 3, \ldots$

Thus, using Assumption (iv) with inequalites (4.10) and (4.12), we have, that $\{x_i\}$, $i = 0, 1, 2, \ldots$. is a Cauchy sequence and there is a limit $\tilde{x}$ of that sequence. Using similar argument as in the Theorem 5 we get the following estimation

$$\sup_{t\in\mathcal{R}} \sqrt{E\left\{|\tilde{x} - x_0|^2\right\}} \leq (1 - \alpha r)^{-1} \sup_{t\in\mathcal{R}} \rho(l) \tag{4.13}$$

since $\alpha r < 1$. This argument ends the proof.

5. Methods of statistical linearization

- METHOD 1. The first SLM replaces the nonlinear function $f(t,x)$, $f: \mathcal{R} \times \mathcal{R}^n \to \mathcal{R}^n$, by a n-dimensional linear function

$$l(t,y) = C(t)y_1 + c(t)$$

where $y = [y_1, y_2]^\top \in \mathcal{R}^n$, $y_1 \in \mathcal{R}^m$, $m \leq n$, that minimizes, over all matrices $C(\cdot)$ and $c(\cdot)$ the mean-square difference

$$\rho_g(l) := E\Big\{|f(t,x(t,\omega)) - l(t,x(t,\omega))|^2\Big\}$$

for each $t \in \mathcal{R}$ and for a certain class of stochastic processes.

- LEMMA 7. The functional $\rho_g(l)$ takes a minimal value, for each $t \in \mathcal{R}$, for the following values of coefficients:

$$\begin{aligned}
&(i) \quad \tilde{C}_1(t) = \Gamma_1(t)K^{-1}(t)\\
&(ii) \quad \tilde{c}_1(t) = E\Big\{f(t,x(t,\omega)) - \tilde{C}_1(t)E\{x(t,\omega)\}\Big\}\\
&(iii) \quad \Gamma_1(t) = E\Big\{f(t,x(t,\omega))[x(t,\omega) - E\{x(t,\omega)\}]^\top\Big\}\\
&(iv) \quad K(t) = E\Big\{[x(t,\omega) - E\{x(t,\omega)\}][x(t,\omega) - E\{x(t,\omega)\}]^\top\Big\}
\end{aligned}$$

The minimal value of the functional appears for the coefficients

$$\tilde{l}(t,y) = \tilde{C}_1(t)y_1 + \tilde{c}_1(t)$$

Proof. It is given by Bunke (1972).

- METHOD 2. Consider, similarly as above, the nonlinear function $f(t,x(t,\omega))$ and the linear approximation $l(t,x(t,\omega))$. Notion is the same as in Method 1. The idea of considered method is well known (cf Kazakov (1975); Krasovskiĭ (1974); Pugachev (1962)). We are looking for such a linear function $l(t,x(t,\omega))$ of the random value $x(t,\omega)$, which has a mean value and a covariance matrix equal to those of $f(t,x(t,\omega))$, for each $t \in \mathcal{R}$. We get the following Lemma.

• LEMMA 8. Assume, the following conditions are satisfied:

$$
\begin{aligned}
&(i) \quad \tilde{C}_2(t) = \Gamma_2^{1/2}(t)K^{-1/2}(t)\\
&(ii) \quad \tilde{c}_2(t) = E\Big\{f(t,x(t,\omega)) - \tilde{C}_2(t)E\{x(t,\omega)\}\Big\}\\
&(iii) \quad \Gamma_2(t) = E\Big\{\Big[f(t,x(t,\omega)) - E\{f(t,x(t,\omega))\}\Big]\cdot\\
&\qquad\qquad \cdot\Big[f(t,x(t,\omega)) - E\{f(t,x(t,\omega))\}\Big]^{\mathsf{T}}\Big\}\\
&(iv) \quad K(t) = E\Big\{\Big[x(t,\omega) - E\{x(t,\omega)\}\Big]\Big[x(t,\omega) - E\{x(t,\omega)\}\Big]^{\mathsf{T}}\Big\}
\end{aligned}
$$

then the expected value and the covariance matrices of the linear function $l(\cdot,\cdot)$ and the nonlinear function $f(\cdot,\cdot)$ are equal.

Proof. It is given by Skrzypczyk (1994a).

Remark. Generally the presented methods are very close to each other, and in the example only the first method is considered.

• METHOD 3. Skrzypczyk (1994) and Zhang et al. (1991) propose a new linearization technique, namely, the mean-square value of the difference of the potential energies, associated with the original nonlinear equation, denoted as $\mathcal{U}(\cdot)$, and its equivalent linear counterpart, is to be minimized. Below the general scheme for multidimensional systems is presented. Assume that nonlinearity $f(\cdot)$ is time-invariant. For simplicity we denote that $\mathcal{U}(0) = 0$.

In the proposed linearization scheme, we require that

$$
\begin{aligned}
\rho &:= E\Big\{\Big|\mathcal{U}(x(t,\omega)) - \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}x_i(t,\omega)x_k(t,\omega)\Big|^2\Big\} =\\
&= E\Big\{\Big|\mathcal{U}(x(t,\omega)) - \frac{1}{2}\sum_{i=1}^{n} x^{\mathsf{T}}(t,\omega)C_i x_i(t,\omega)\Big|^2\Big\} = \min
\end{aligned}
\tag{5.1}
$$

where $\mathcal{U}(\cdot)$ denotes the potential energy corresponding to the nonlinearity $f(\cdot)$ and

$$
\begin{aligned}
C &= [c_{ik}] = [C_1, C_2, ..., C_n]\\
C_k &= [c_{k1}, c_{k2}, ..., c_{kn}]^{\mathsf{T}}
\end{aligned}
$$

for $i,k = 1,2,...,n$. Let

$$
\nabla_x = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right]^{\mathsf{T}}
$$

This is accomplished by the condition

$$\nabla_{C_k}\rho = E\Big\{\Big[\mathcal{U}(x(t,\omega)) - \frac{1}{2}\sum_{i=1}^{n} x^{\mathsf{T}}(t,\omega)C_i x_i(t,\omega)\Big]x(t,\omega)x_k(t,\omega)\Big\} = 0 \tag{5.2}$$

$$2\Gamma_k = \sum_{i=1}^{n}\phi_{ki}C_i$$

where we denote

$$\Gamma := [\Gamma_1, \Gamma_2, ..., \Gamma_n] \qquad \Gamma_k := E\Big\{\mathcal{U}(x(t,\omega))x(t,\omega)x_k(t,\omega)\Big\}$$

$$\Phi := [\phi_{ki}] \qquad \phi_{ki} := E\Big\{x_k(t,\omega)x_i(t,\omega)x(t,\omega)x^{\mathsf{T}}(t,\omega)\Big\}$$

for $i,k = 1,2,...,n$. Finally we get

$$2\Gamma = \Phi C \tag{5.3}$$

$$C = 2\Phi^{-1}\Gamma \tag{5.4}$$

Eq (5.4) ends the considerations.

Skrzypczyk (1994) and Zhang et al. (1991) suggest that the new method of statistical linearization turns out to be superior to conventional linearization methods 1st and 2nd, for some systems with hardening nonlinearities.

6. Example

To illustrate considerations on accuracy of statistical linearization and to compare approximations we consider a one-dimensional vibrating system, whose motion is described by a differential equation in the normalized form

$$\ddot{x}(t) + \beta\dot{x}(t) + F(x(t)) = z(t) \qquad t \in \mathcal{R}^1 \tag{6.1}$$

where $(\cdot) = \frac{d}{dt}$, $\beta = \text{const} > 0$ denotes a coefficient of linear viscous damping, and the function $F(x)$, $x \in \mathcal{R}^1$ represents the characteristic of the nonlinear elastic force. It is assumed further that $F(\cdot)$ takes the form

$$F(x) = \begin{cases} \gamma(x-1)+1 & \text{for} \quad x \le -1 \\ x & \text{for} \quad -1 < x < 1 \\ \gamma(x+1)-1 & \text{for} \quad x \ge 1 \end{cases}$$

and two cases are considered $\gamma = 0.5$ and $\gamma = 2.0$.

We consider the case when the excitation force is a stationary 2nd order stochastic process with a mean value equal to zero and a spectral density of the form

$$S_z(iu) = \frac{S_0}{1 + u^2\tau^2} \qquad u \in \mathcal{R}^1 \tag{6.2}$$

where $S_0 > 0$ and τ are some constants. The resulting density function $p(\cdot,\cdot,\cdot)$ is easy obtainable following considerations given by Skrzypczyk (1993) and (1994b) and has the form

$$p(y_1, y_2, y_3) = p^{(3)}(y_1, y_2, y_3) = N \exp\Big[-\frac{2\tau}{S_0}\int_0^{y_1}\Big(\frac{dF}{dz} + \frac{\beta}{\tau}\Big)F(z)\,dz + \tag{6.3}$$

$$-\frac{\beta}{S_0}\Big(1 + \beta\tau + \tau^2\frac{dF}{dy_1}\Big)y_2^2 - \frac{\tau(1+\beta\tau)}{S_0}y_3^2 - \frac{2\tau F(y_1)}{S_0}y_3\Big]$$

where $y_1 = x$, $y_2 = \dot{x}$, $y_3 = \ddot{x}$ and N is a constant, which is found from the normalization condition

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p^{(3)}(y_1, y_2, y_3)\,dy_1dy_2dy_3 = 1$$

We recall that the one-dimensional probability density of $y_1 = x$ is

$$p^{(1)}(x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p^{(3)}(y_1, y_2, y_3)\,dy_2dy_3$$

and the exact mean-square displacement

$$\sigma_x^2 = \int_{-\infty}^{+\infty} x^2 p^{(1)}(x)\,dx$$

can be found analytically or numerically.

A linearized form corresponding to Eq (6.1) is a linear, stochastic differential equation

$$\ddot{y}(t) + \beta\dot{y}(t) + C_i y(t) = z(t) \qquad t \in \mathcal{R}^1 \tag{6.4}$$

and a value of C_i, $i = 1, 2, 3$ depends on the choice of a linearization method.

We denote by $\widetilde{x}(\cdot)$ and $\widetilde{y}_i(\cdot)$ R-solutions (cf Bunke (1972)) of Eqs (6.1) and (6.4) respectively and by σ_x, $\sigma_{y,i}$ their corresponding standard deviations. Further we use the notation σ_0 for a standard deviation of a solution to a linear version of Eq (6.1) i.e. for $\gamma = 1.0$. We discuss absolute standard deviation errors

$$\sigma_{err,i} = |\sigma_x - \sigma_{y,i}|$$

as well as relative errors

$$\eta_i = \sigma^2_{err,i}/\sigma^2_0$$

and $\sigma^2_{y,i}/\sigma^2$ associated with the conventional $(i = 1,2)$ and the new $(i = 3)$ SLM's. For simplicity only stationary solutions are considered.

The exact mean-square displacements are computed numerically according to Eq (6.3) for a nonlinear case, as well as for a linear one. Corresponding numerical values, for discussed SLM's, follow the results of Section 5.

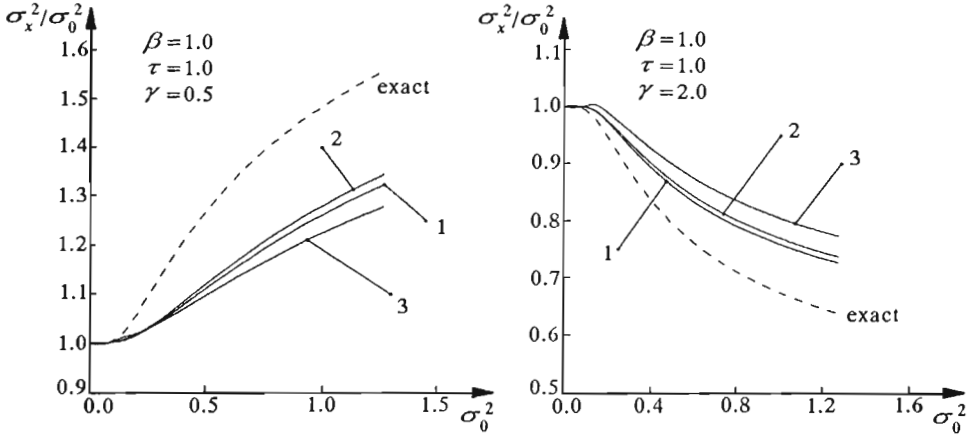

Fig. 1. Comparison of relative mean square displacements in the nonlinear oscillator via stochastic linearization with the exact solution (curve i – SLM of ith type)

Fig.1 contrasts relative values of mean square displacement evaluated exactly, along with the results furnished by the three different SLM's. Curve i is associated with the ith SLM, compare notion in Section 5. Comparison between the exact solution and the SLM was performed earlier by Skrzypczyk (1994a) and Zhang et al. (1991). Fig.2 depicts relative errors η_i associated with the ith SLM, respectively. As is seen from Fig.1 and Fig.2, the conventional linearization methods $(i = 1,2)$ yield results which are closer to the exact solution, than those calculated on the basis of the new one $(i = 3)$.

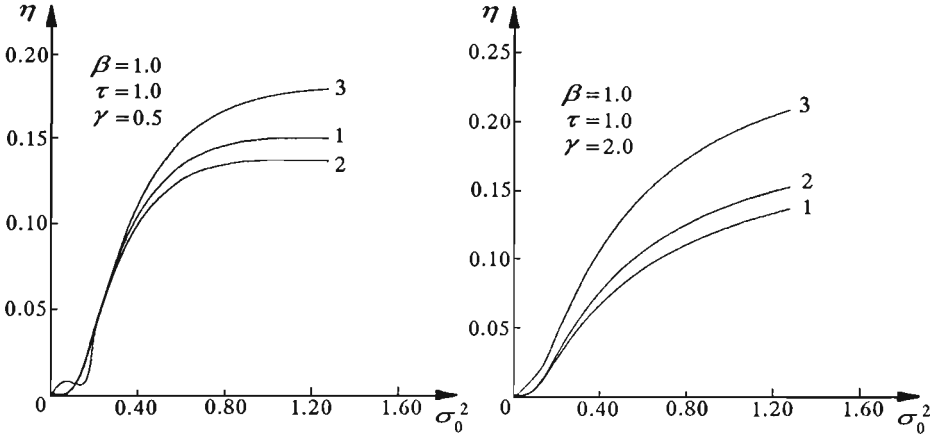

Fig. 2. Comparison of relative standard deviation errors for different SLM's (curve i – SLM of ith type)

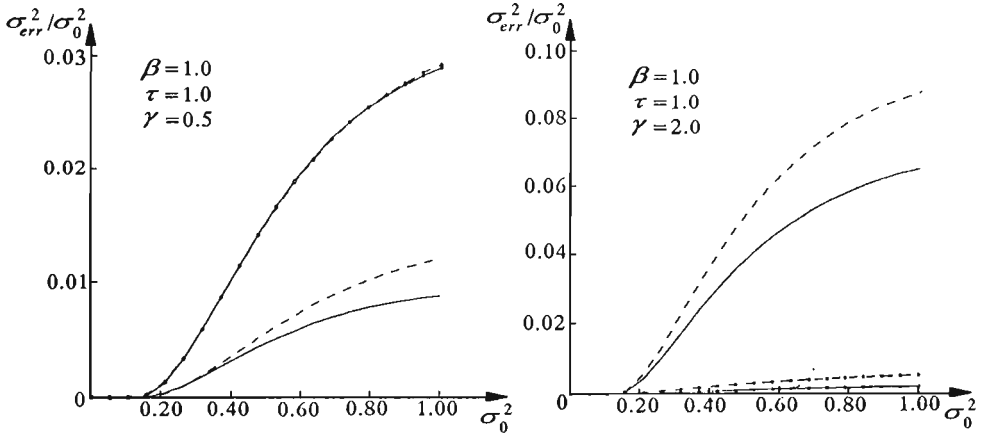

Fig. 3. Comparison of absolute standard deviation errors in the nonlinear oscillator via stochastic linearization with theoretical bounds (continuous line – exact error of a conventional SLM of type 1st, dashed line – exact error of a new SLM of type 3, lines with centered symbols – theoretical bounds of errors of associated SLM's)

Following results of Theorem 6 we get theoretical bounds for standard deviation errors since

$$\sup_{t\in\mathcal{R}} |\sigma_x - \sigma_y| \leq \sup_{t\in\mathcal{R}} \sqrt{E\left\{|\tilde{x}(t,\omega) - \tilde{y}(t,\omega)|^2\right\}} \leq \text{const} \sup_{t\in\mathcal{R}} \rho(l)$$

and the constant is given by Eq (6.4). Further the conventional SLM's ($i = 1,2$) are not distiguished, since results are very close to each other.

Fig.3 portrays the absolute standard deviation errors $|\sigma_x - \sigma_y|$, for two linearization methods (1st and 3rd), evaluated exactly, along with theoretical bounds calculated following Eq (6.4).

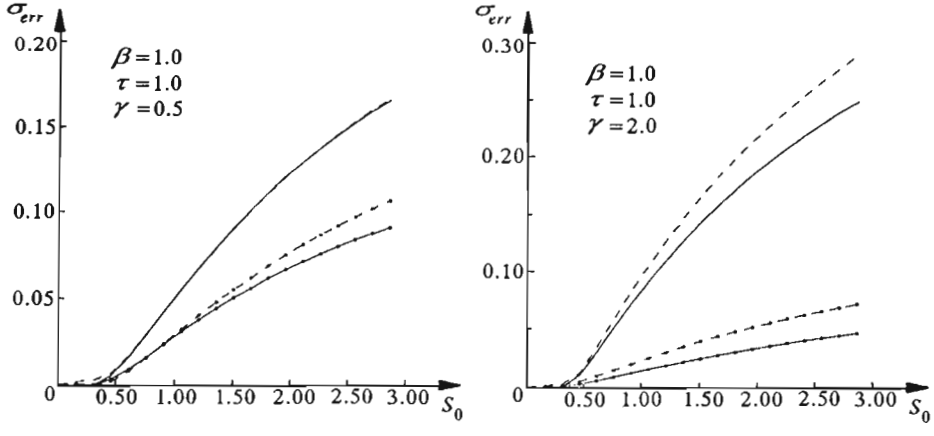

Fig. 4. Comparison of relative mean square displacements errors in the nonlinear oscillator via stochastic linearization with theoretical bounds (continuous line – exact error of a conventional SLM of type 1st, dashed line – exact error of a new SLM of type 3, lines with centered symbols – theoretical bounds of errors of associated SLM's)

The relative standard deviation errors $(\sigma_x - \sigma_y)^2/\sigma_0^2$, for the same linearization methods, evaluated exactly, as well as theoretical bounds, are presented in Fig.4.

7. Conclusions

The main aim of above considerations it is not only to contrast three, actually used, methods of statistical linearization technique, applied to multidegree-of-freedom stochastic systems, but to give a modern error theory

of used methods and to obtain bounds of the error between the exact and the approximate linearized solutions. Proposed mathematical technique is strictly related to stability analysis of nonlinear stochastic systems and gives results of similar accuracy. To obtain more accurate error bounds, other, more subtle techniques must be used.

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Analiza dokładności metod linearyzacji statystycznej w ciągłych układach nieliniowych opisanych losowymi równaniami całkowymi

Streszczenie

Celem pracy jest zbadanie dokładności przybliżonych metod analizy, opartych na koncepcji linearyzacji statystycznej, układów dynamicznych opisanych nieliniowymi równaniami całkowymi Volterry drugiego rodzaju. Rozpatrywane są zarówno układy dynamiczne niestacjonarne jak i stacjonarne. Technika analizy harmonicznej nieprzemiennych operatorów przyczynowych z przestrzeni $L^1(\mathcal{R}; L(\mathcal{R}^n, \mathcal{R}^n))$ zastosowana

została do analizy specjalnych algebr Banacha operatorów całkowych, które są tzw. filtrami granicznymi. Otrzymano, ścisle z matematycznego punktu widzenia, oszacowanie błędów analizy nieliniowych układów dynamicznych wynikających z zastosowania różnych metod linearyzacji statystycznej. Podano oszacowanie błędu, wynikającego z linearyzacji statystycznej, pomiędzy rozwiązaniami: dokładnym i przybliżonym problemów nieliniowych. Przeanalizowano również warunki istnienia, ciągłego w sensie średniokwadratowym, ścisłego rozwiązania stacjonarnego. Dla porównania przedstawionych metod z wynikami ścisłymi, dla różnych metod linearyzacji statystycznej, dokonano analizy prostego układu dynamicznego z wymuszeniem, które jest stacjonarnym procesem stochastycznym.

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