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NONLINEAR MACRO-MICRO DYNAMICS OF LAMINATED STRUCTURES

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The aim of this contribution is to formulate and investigate a model of the periodically laminated medium made of two isotropic highly-elastic constituents. The main feature of the proposed approach is that the resulting relations depend on the microstructure length parameter and hence describe dispersion phenomena and higher free vibration frequencies in the dynamic behaviour of a micro-laminated body. The obtained model will be used to the dynamic stability and the wave propagation problem analysis.

1. Introduction

The linear elastodynamics of periodic composite materials, which takes into account the effect of microstructure length dimension on the dynamic behaviour of the medium, was developed by Woźniak (1993c) and Woźniak et al. (1993) as the refined macro-dynamics of periodic structures and then applied in a series of papers (cf Woźniak (1993a,b), Wierzbicki (1993), Mazur-Śniady (1993)) to the analysis of special problems. In this paper there are considered laminated bodies made of highly-elastic isotropic constituents, which in the reference configuration of a body have a periodic structure. For the sake of simplicity we restrict ourselves to composite bodies made of two different elastic materials but more general case can be also described by the proposed approach. We also assume the perfect bonding between the adjacent laminae and homogeneity of constituents related to the reference configuration in which the laminated body has a periodic material structure. It has to be remembered that under arbitrary finite deformations the observed material structure of the composite can be no longer periodic and the strain energy function of every constituent (per unit volume of the deformed body) depends on a position of the material particle under consideration. However, in the engineering problems concerning highly deformed elastic laminates, we usually deal with the class of deformations for which the adjacent repetitive material cells of the composite (which in the reference configuration coincide with a certain representative volume element of the periodic structure) undergo only slightly different strain distributions. It means that, roughly speaking, the small fragments of the deformed composite (made of a few adjacent repetitive material cells) behave as "nearly periodic" in the considered dynamic processes. This fact will be a basis of the modelling approach proposed in this paper and will make it possible to apply certain procedures of modelling, typical for the refined linear-elastodynamics of periodic structures, to the macro-modelling of highly elastic laminates. The characteristic feature of the proposed below macro-modelling procedure is that it retains length-dimension parameter of a laminae and hence the resulting relations also involve terms describing the macro-inertial properties of a composite body. Such situation does not arise in the known asymptotic approaches to the modelling of micro-periodic composites, where the resulting equations represent a certain homogenized model of a periodic material independent of the microstructure length-dimension parameter (cf Francfort and Marat (1992), Boutin and Auriault (1993), Cherkaev (1993), Wągrowska (1986) and (1988), Matysiak and Nagórko (1989), Nagórko (1989), Kaczyński and Matysiak (1988)). Since in the proposed approach this length parameter is retained then the obtained equations will be referred to as the equations of the macro-micro dynamics of laminated materials.

Denotations. By $0x_1x_2$ we denote the orthogonal cartesian coordinate system in the physical space; mathematical objects related to this system are endowed with the latin sub- and superscripts $i, j, \ldots$ running over $1, 2, 3$. The material coordinates of particles will be denoted by X^α and hence the greek sub- and superscripts α, β,... are related to the material coordinate system, which in the reference configuration of the body is assumed to coincide with the cartesian orthogonal system. In the motion of the body X^α are treated as the convective coordinates. Summation convention holds for all kind of indices. The subscript R is used in order to underline the fact that the corresponding mathematical object is a density related to the unit volume of the body in its reference configuration. The region occupied by this body in the reference configuration will be denoted by B_R and the points of this region by $\boldsymbol{X}$, $\boldsymbol{X} \in B_R$, where $\boldsymbol{X} = (X^1, X^2, X^3) = (X^\alpha)$. By t we denote time coordinate. The deformation function of the body at a time t, $t \in [t_0, t_f]$, will be denoted by

$$x_i = \chi_i(\boldsymbol{X}, t) \qquad\qquad \boldsymbol{X} = (X^\alpha) \in B_R \tag{1.1}$$

and hence the covariant components of the metric deformation tensor are

$$c_{\alpha\beta} = \chi_{i,\alpha}\chi_{i,\beta} \tag{1.2}$$

while contravariant components are $c^{\alpha\beta}$, where $c^{\alpha\beta}c_{\beta\gamma} = \delta^\alpha_\gamma$. The body forces (per mass unit) will be denoted by b_i and are assumed to be constant. The boundary surface tractions, related to the reference configuration are denoted by t_{Ri} and to the actual configuration, described by the convective coordinate system, by t^α, respectively.

2. Modelling hypotheses

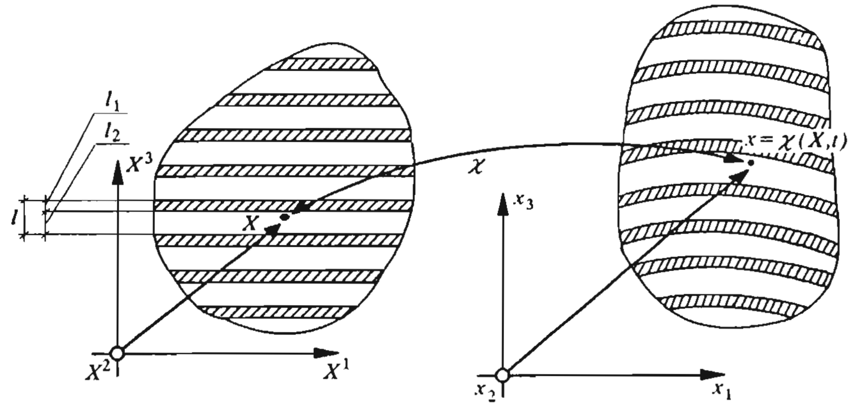

Fig. 1. Reference and deformed configurations of a certain fragment of the laminated body

The object of considerations is a two-constituent laminated body which in the reference configuration has a periodic material structure. For the time being we assume that laminae interfaces are normal to X^3-axis. The reference and the deformed configurations, respectively, of a certain fragment of this body are shown in Fig.1. It is assumed that the smallest characteristic length dimension of the whole region B_R, occupied by the body in the reference configuration, is sufficiently large compared to the thickness l of the representative two component layer of the laminate in its periodic reference state, cf Fig.1. Parameter l will be treated as the microstructure length parameter. Every basic layer of the laminate is made of two homogeneous sublayers having thicknesses l', l'' and representing two different materials with constant mass densities ρ'_R, ρ''_R and strain energy functions $W'_R(\boldsymbol{F})$, $W''_R(\boldsymbol{F})$ (depending on the deformation gradient $\boldsymbol{F}$) related to the specific volume of the reference configuration.

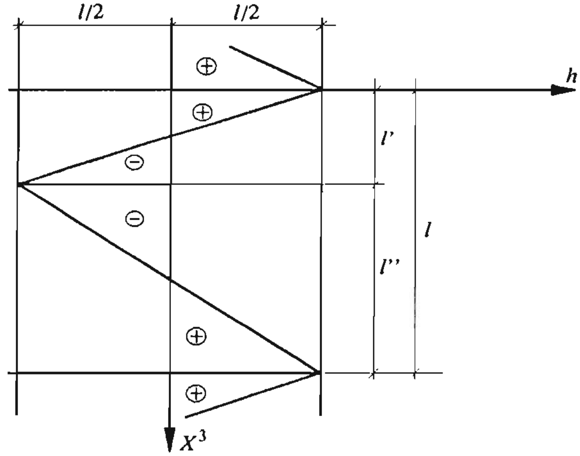

Fig. 2. Diagram of the micro-shape function in $[0, l]$

In order to formulate the modelling hypotheses we adapt the general line of approach given by Woźniak (1993c) to the highly deformed materials. To this end we introduce what is called microshape function $h = h(X^3)$, $X^3 \in R$, which is l-periodic and continuous; the diagram of this function in $[0, l]$ is shown in Fig.2. Let us observe, that $h(X^3) \in \mathcal{O}(l)$ and $h_{,3}(X^3) \in \mathcal{O}(1)$. We also apply the concept of a macro-function (related to the l-periodic material structure of B_R) $F(\cdot)$, which is real valued, defined on B_R and satisfies conditions

$$(*) \qquad \forall (X, Z) \in B_R \quad \|X - Z\| < l \rightarrow |F(X) - F(Z)| < \lambda_F$$

where λ_F is a certain small numerical-approximation parameter related to calculations of a function F. In the sequel we shall deal with regular macro-functions, which can also depend on t and have to satisfy condition of the form $(*)$ together with all their derivatives, including time derivatives. Using the concept of a macro-function we shall apply the following formula for calculation of integrals over B_R, involving l-periodic function $f(X^3)$ and macro-function $F(X)$

$$(**)\qquad \int_{B_R} f(X^3)F(X)\,dV_R = <f> \int_{B_R} F(X)\,dV_R + \mathcal{O}(\lambda_F)$$

where

$$<f> = \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} f(X^3)\,dX^3$$

which is the averaged constant value of $f(\cdot)$. Moreover, since $h(X^3) \in \mathcal{O}(l)$ and $F_{,\alpha}(X) \in \mathcal{O}(\lambda/l)$, then

$$(***)\qquad \Big(h(X^3)F(X)\Big)_{,\alpha} = h_{,3}(X^3)\delta_{\alpha 3}F(X) + \mathcal{O}(\lambda_F)$$

where $h(X^3)$ is the microshape function introduced above.

The first modelling hypothesis is the *kinematic modelling hypothesis* which restricts the class of all possible deformations of a laminated body to the deformations given by

$$\chi_i = \chi_i(X,t) = P_i(X,t) + h(X^3)Q_i(X,t) \qquad X \in B_R \tag{2.1}$$

where $P_i(X,t)$, $Q_i(X,t)$ are regular macro-functions. The function $P_i(X,t)$ determines what are called macro-deformations of the laminated body while the second term in Eq (2.1) describes the expected form of micro-disturbances caused by the micro-inhomogeneity of this body. Functions $Q_i(X,t)$ will be called inhomogeneity correctors. Let us observe, that under denotations: $\phi' \equiv -l/l'$, $\phi'' \equiv -l/l''$, the deformation gradient will be given by

$$\chi_{i,\alpha}(X,t) = P_{i,\alpha}(X,t) + \delta_{\alpha 3}\left\{ \begin{array}{c} \phi' \\ \phi'' \end{array} \right\} Q_i(X,t) + \mathcal{O}(\lambda_Q) \qquad X \in B_R$$

where ϕ', ϕ'' are related to the deformation gradients in laminae with thicknesses l', l'', respectively. Denoting $N' \equiv \phi'\delta_{\alpha 3}$, $N'' \equiv \phi''\delta_{\alpha 3}$, we obtain the alternative form of the above equation

$$\chi_{i,\alpha}(X,t) = P_{i,\alpha}(X,t) + \left\{ \begin{array}{c} N'_a \\ N''_a \end{array} \right\} Q_i(X,t) + \mathcal{O}(\lambda_Q) \qquad X \in B_R \tag{2.2}$$

which also holds in an arbitrary system of material coordinates. Hence in the sequel planes $X^3 =$const may be not parallel to the laminae interfaces in the reference configuration.

The second modelling hypothesis will be referred to as the *macro-approximation hypothesis* and states that in calculations of the global energy terms $\mathcal{O}(\lambda_F)$ in formulas of the form (**), (* * *) can be neglected (F stands for an arbitrary macro-function).

It is easy to see that this hypothesis is strictly related to the concept of a macro-function and to the class of deformations given by Eq (2.1).

3. Modelling approach

The passage from micro- to macro- mechanics for the bodies under consideration will be based on the assumption that $P_i(X,t)$, $Q_i(X,t)$ are independent dynamic variables. We shall use the principle of stationary action $\delta A = 0$, where

$$A = \int_{t_0}^{t_1} (K - P - W)\, dt \tag{3.1}$$

is the action functional where K, P are kinetic and potential energies, respectively, W is the energy of external loadings, given by

$$\begin{gathered} K = \frac{1}{2}\int_{B_R} \rho_R(X)\dot{\chi}_i\dot{\chi}_i\, dV_{R'} \qquad P = \int_{B_R} \varepsilon_R(X,\nabla\chi)\, dV_{R'} \\ W = \int_{B_R} \rho_R(X) b_i\chi_i\, dV_{R'} + \oint_{\partial B_R} t_{Ri}\chi_i\, dA_R \end{gathered} \tag{3.2}$$

where $\varepsilon_R(X,\nabla\chi)$ is the strain energy function given by $\varepsilon'_R(\nabla\chi)$ and $\varepsilon''(\nabla\chi)$ in both material constituents of the laminate, respectively.

The modelling approach consists of:

(i) Substituting the right-hand sides of Eq (2.1) into Eqs (3.2)

(ii) Calculations of integrals in Eq (3.2) by using formulae (**), (* * *); setting $\sigma' \equiv l'/l$, $\sigma'' \equiv l''/l$, after some transformations we obtain

$$K = \frac{1}{2}\int_{B_R} \Big(<\rho_R> \dot{P}_i\dot{P}_i\, dV_R + <\rho_R h^2> \dot{Q}_i\dot{Q}_i \Big)\, dV_R + \mathcal{O}(\lambda_{\dot{P}}) + \mathcal{O}(\lambda_{\dot{Q}})$$

$$
\begin{aligned}
P &= \int\limits_{B_R} \Big[\sigma'\varepsilon'(\nabla P + \boldsymbol{N}' \otimes \boldsymbol{Q}) + \sigma''\varepsilon_R''(\nabla P + N'' \otimes \boldsymbol{Q})\Big]\, dV_R + \\
&+ \mathcal{O}(\lambda_{\nabla P}) + \mathcal{O}(\lambda_Q)
\end{aligned}
\tag{3.3}
$$

$$
W = \int\limits_{B_R} <\rho_R> b_i P_i \, dV_R + \int\limits_{\partial B_R} t_{Ri} P_i \, dA_R + \mathcal{O}(\lambda_P)
$$

where we have taken into account Eq (2.2) together with $<\rho_R h>= 0$ and we have neglected terms

$$
\int\limits_{\partial B_R} t_{Ri} h Q_i \, dA_R
$$

which is a certain extra condition imposed both on ∂B_R and t_{Ri}

(iii) Neglecting in Eqs (3.3) terms $\mathcal{O}(\lambda_F)$ by using the macro-approximation hypothesis

(iv) Applying the principle of stationary action δA to the functional (3.1) given by the formulae (3.3) (in which terms $\mathcal{O}(\lambda_F)$ are neglected) which leads to the Euler-Lagrange equations for macro-deformations $P_i(\boldsymbol{X}, t)$ and correctors $Q_i(\boldsymbol{X}, t)$.

Under denotation

$$
<\varepsilon_R> (\nabla P, \boldsymbol{Q}) \equiv \sigma'\varepsilon'(\nabla P + \boldsymbol{N}' \otimes \boldsymbol{Q}) + \sigma''\varepsilon_R''(\nabla P + N'' \otimes \boldsymbol{Q})
$$

we obtain the Euler-Lagrange equation in the form

$$
\begin{aligned}
&\Big(\frac{\partial <\varepsilon_R>}{\partial P_{i,\alpha}}\Big)_{,\alpha} - <\rho_R> \ddot{P}_i + <\rho_R> b_i = 0 \\
&<\rho_R h^2> \ddot{Q}_i + \frac{\partial <\varepsilon_R>}{\partial Q_i} = 0
\end{aligned}
\tag{3.4}
$$

which holds for every $\boldsymbol{X} \in B_R$, and the natural boundary conditions on B_R

$$
\frac{\partial <\varepsilon_R>}{\partial P_{i,\alpha}} n_{R\alpha} = t_R^i \tag{3.5}
$$

where $n_{R\alpha}$ is the unit outward normal to ∂B_R. Since Eqs (3.4) and (3.5) involve exclusively macro-functions, being independent of any highly-oscillating function describing the material properties of a laminated medium, then they represent a certain macro-model of the body under consideration which can be used in engineering applications of the theory. The microstructure properties of this composite are given by the micro-inertial modulus

$$<\rho_R h^2>= \frac{l^2}{12}(\sigma'\rho' + \sigma''\rho''_R) = \frac{l^2}{12} <\rho_R> \tag{3.6}$$

which depends on the square of the microstructure length parameter l. In the subsequent section we shall transform the obtained results to the form typical for continuum mechanics.

4. Governing relations

Introducing the following relations

$$S^{i\alpha}_R = \frac{\partial <\varepsilon_R>}{\partial P_{i,\alpha}} \qquad\qquad H^i_R = \frac{\partial <\varepsilon_R>}{\partial Q_i} \tag{4.1}$$

we can rewrite Eqs (3.4) to the form

$$\begin{aligned} &S^{i\alpha}_{R,\alpha} - <\rho_R> \ddot{P}^i + <\rho_R> b^i = 0 \\ &<\rho_R h^2> \ddot{Q}^i + H^i_R = 0 \end{aligned} \tag{4.2}$$

and Eq (3.5) will be given by

$$S^{i\alpha}_R n_{R\alpha} = t^i_R \tag{4.3}$$

Object with components $S^{i\alpha}_R$ will be referred to as the first Piola-Kirchhoff macro-stress tensor and vector H^i_R will be called the micro-dynamical force (related to the reference configuration). These objects have a simple interpretation. Bearing in mind that

$$\begin{aligned} {}'t^i_R &= \frac{\partial \varepsilon'_R}{\partial \chi_{i,\alpha}} = \frac{\partial \varepsilon'_R}{\partial P_{i,\alpha}} \\ {}''t^i_R &= \frac{\partial \varepsilon''_R}{\partial \chi_{i,\alpha}} = \frac{\partial \varepsilon''_R}{\partial P_{i,\alpha}} \end{aligned} \tag{4.4}$$

are the first Piola-Kirchhoff stress tensors in both constituents, respectively (which can be called micro- or partial-stress tensors) and

$$\frac{\partial \varepsilon'_R}{\partial Q_i} = \frac{\partial \varepsilon'_R}{\partial P_{i,\alpha}} N'_\alpha \qquad\qquad \frac{\partial \varepsilon''_R}{\partial Q_i} = \frac{\partial \varepsilon''_R}{\partial P_{i,\alpha}} N''_\alpha$$

from Eqs (4.1) and definitions of N_α we obtain the following interrelations between the Piola-Kirchhoff micro-stress tensors and macro-stresses as well as micro-dynamical force

$$\begin{aligned} S_R^{i\alpha} &=' t_R^{i\alpha}\sigma' +'' t_R^{i\alpha}\sigma'' \\ H_R^i &=' t_R^{i\alpha} N'_\alpha \sigma +'' t_R^{i\alpha} N''_\alpha \sigma'' \end{aligned} \tag{4.5}$$

If X^3-axis is normal to the laminae interfaces in the reference configuration, then by means of $N'_\alpha\sigma' + N''_\alpha\sigma'' = 0$ we obtain $H_R^i =' t_R^{i3} -'' t_R^{i3}$. Let us observe that if in Eqs (4.2) the micro-inertial term $< \rho_R h^2 > \ddot{Q}_i$ will be neglected (as being of an order $O(l^2)$, cf Eq (3.6)) then $H_R^i = 0$, i.e., the micro-dynamical force is equal to zero. The condition $H_R^i = 0$ holds in quasi-stationary problems and in the asymptotic approximation theory, where the microstructure of the laminate is scaled down by assuming $l \searrow 0$ (cf Francfort and Marat (1992), Boutin and Auriault (1993), Cherkaev (1993), Wągrowska (1986) and (1988), Matysiak and Nagórko (1989), Nagórko (1989), Kaczyński and Matysiak (1988)). In these cases, taking into account the above formula for H_R^i, on the laminae interfaces we obtain the stress continuity conditions $S_R^{i3} =' t_R^{i3} ='' t_R^{i3}$. Eqs (4.1) ÷ (4.3) constitute the governing equations of what will be called the nonlinear macro-microdynamics of highly-elastic laminates or the nonlinear refined macro-dynamics of laminates. Similarly to Woźniak (1993c), the term "refined" is related to the presence in the above equations of the microinertial modulus given by Eq (3.6) depending on the microstructure length parameter l. The equations of the nonlinear refined macrodynamics (similarly to those of the linear theory, Woźniak (1993c)) involve the constitutive relations (4.1), the equations of motions (4.2) and natural boundary conditions (4.3), respectively. It has to be emphasized that the unknown fields $P_i(\boldsymbol{X},t)$, $Q_i(\boldsymbol{X},t)$, $\boldsymbol{X} \in B_R$, $t \in [t_0, t_f]$, satisfying these equations have a physical sense only if they are macro-functions (related to the l-periodic material structure of B_R). The general discussion of the above obtained governing relations of the non-linear refined macrodynamics is analogous to that of the linear refined macrodynamics; for details the reader is referred to Woźniak (1993c). Let us transform Eqs (4.1) ÷ (4.3) to the alternative form bearing in mind that

$$\varepsilon'_R(\nabla\chi) = U'_R(\boldsymbol{c}) \qquad\qquad \varepsilon''_R(\nabla\chi) = U''_R(\boldsymbol{c})$$

where $\boldsymbol{c}$ is a metric deformation tensor given by Eq (1.2). Taking into account Eq (2.2) we obtain

$$\left\{ \begin{array}{c} c'_{\alpha\beta} \\ c''_{\alpha\beta} \end{array} \right\} = C_{\alpha\beta} + 2 \left\{ \begin{array}{c} N'_{\alpha} \\ N''_{\alpha} \end{array} \right\} Q_{\beta} + \left\{ \begin{array}{c} N'_{\alpha} N'_{\beta} \\ N''_{\alpha} N''_{\beta} \end{array} \right\} Q^2 \tag{4.6}$$

where

$$C_{\alpha\beta} \equiv P_{i,\alpha} P_{i,\beta} \qquad Q_{\beta} \equiv P_{i,\beta} Q_i \qquad Q^2 \equiv Q_i Q_i$$

In the sequel the strain energy functions will be assumed in the form $U' = U'_R < \rho_R >^{-1}$, $U'' = U''_R < \rho_R >^{-1}$, which yields

$$< U > (C, Q, Q^2) = \sigma' U'(c') + \sigma'' U''(c'') \tag{4.7}$$

where $\boldsymbol{c}'$ and $\boldsymbol{c}''$ are given by Eq (4.6). Defining

$$J \equiv \det \nabla \boldsymbol{P} \qquad p \equiv J^{-1} < \rho_R > \qquad \mu \equiv J^{-1} < \rho_R h^2 > \tag{4.8}$$

and introducing matrix Ξ_i^{α} inverse to $P_{i,\alpha}$, we shall transform Eqs (4.1) $\div$ (4.3) to the convective coordinate form. The convective macro-stress tensor and the micro-dynamical force related to the actual configuration of the body (at time t) will be given by

$$S^{\alpha\beta} \equiv J^{-1} S_R^{i(\alpha} \Xi_i^{\beta)} \qquad H^{\alpha} \equiv J^{-1} H_R^i \Xi_i^{\alpha} \tag{4.9}$$

The constitutive equations in $S^{\alpha\beta}$, H^{α} are

$$\begin{aligned} S^{\alpha\beta} &= \rho \Big(2 \frac{\partial < U >}{\partial C_{\alpha\beta}} + \frac{\partial < U >}{\partial Q_{(\alpha}} Q^{\beta)} \Big) \\ H^{\alpha} &= \rho \Big(\frac{\partial < U >}{\partial Q_{\alpha}} + 2 \frac{\partial < U >}{\partial Q^2} Q^{\alpha} \Big) \end{aligned} \tag{4.10}$$

and the equations of motion take the form

$$\begin{aligned} S^{\alpha\beta}\Big|_{\beta} - \rho \ddot{P}_i \Xi_i^{\alpha} + \rho b^i \Xi_i^{\alpha} &= 0 \\ \mu \ddot{Q}_i \Xi_i^{\alpha} + H^{\alpha} &= 0 \end{aligned} \tag{4.11}$$

where the vertical bar stands for a covariant derivative in the metric $C_{\alpha\beta}$. Eqs (4.10) and (4.11) have to be satisfied in B_t, $B_t \equiv \boldsymbol{P}(B_R, t)$, for every t. The natural boundary conditions are

$$S^{\alpha\beta} n_{\beta} = t^{\alpha} \tag{4.12}$$

where n_β are components of the unit normal vector to ∂B_t and t^α are boundary tractions related to the actual configuration of the body. Eqs (4.10) $\div$ (4.12) represent the convective form of nonlinear refined macrodynamics of the laminated composites under considerations. They have to be considered together with Eqs (4.5) $\div$ (4.8) and constitute the system of governing equations in macro-deformations P_i and correctors Q_i, which are related to the actual configuration of the body.

5. Alternative form of constitutive relations

Substituting the right-hand sides of Eqs (4.4) and (4.5) into definitions (4.9) and introducing strain energies (cf Eq (4.7))

$$U'(\mathbf{c}') = U'_R(\mathbf{c}') <\rho_R>^{-1} \qquad\qquad U''(\mathbf{c}'') = U''_R <\rho_R>^{-1}$$

with $\mathbf{c}'$, $\mathbf{c}''$ given by Eqs (4.6), we obtain

$$\begin{aligned} S^{\alpha\beta} &= 2\rho\Big(\sigma'\frac{\partial U'}{\partial C_{\alpha\beta}} + \sigma''\frac{\partial U''}{\partial C_{\alpha\beta}}\Big) \\ H^\alpha &= 2\rho\Big(\sigma'\frac{\partial U'}{\partial C_{\alpha\beta}}N'_\beta + \sigma''\frac{\partial U''}{\partial C_{\alpha\beta}}N''_\beta\Big) \end{aligned} \tag{5.1}$$

where we have taken into account that

$$\frac{\partial U'}{\partial C_{\alpha\beta}} = \frac{\partial U'}{\partial c'_{\alpha\beta}} \qquad\qquad \frac{\partial U''}{\partial C_{\alpha\beta}} = \frac{\partial U''}{\partial c''_{\alpha\beta}} \tag{5.2}$$

Eqs (5.1) constitute the alternative form of the macro-constitutive relations for the laminates under considerations. Introducing the partial stresses related to the averaged mass density $\rho = J^{-1} <\rho_R>$, given by

$${}'t^{\alpha\beta} \equiv 2\rho\frac{\partial U'}{\partial C_{\alpha\beta}} \qquad\qquad {}''t^{\alpha\beta} \equiv 2\rho\frac{\partial U''}{\partial C_{\alpha\beta}} \tag{5.3}$$

we shall write Eqs (5.1) in the simple form

$$\begin{aligned} S^{\alpha\beta} &= {}'t^{\alpha\beta}\sigma' + {}''t^{\alpha\beta}\sigma'' \\ H^\alpha &= {}'t^{\alpha\beta}N'_\beta\sigma' + {}''t^{\alpha\beta}N''_\beta\sigma'' \end{aligned} \tag{5.4}$$

which corresponds to Eqs (4.5). If X^3-axis is normal to the laminae interfaces in B_R, then $H^\alpha = {}'t^{\alpha 3} - {}''t^{\alpha 3}$.

6. Isotropic laminates

Now assume that both constituents are isotropic. Then

$$U'(\mathbf{c}') = W'(I_1', I_2', I_2') \qquad\qquad U''(\mathbf{c}'') = W''(I_1'', I_2'', I_2'') \tag{6.1}$$

where I_A', I_A'', $A = 1, 2, 3$, are strain invariants which can be assumed in the form

$$I_1 = \mathbf{c}_{\alpha\beta}\delta^{\alpha\beta} = K_1 + J_1$$

$$I_2 = \frac{1}{2}\Big(I_1^2 - \mathbf{c}_{\alpha\beta}\mathbf{c}_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big) = K_2 + K_1 J_1 + J_2$$

$$I_3 = \det \mathbf{c}_{\alpha\beta} = K_3 + K_2 J_1 + K_1 J_2 + J_3$$

where

$$\begin{aligned}
J_1 &\equiv C_{\alpha\beta}\delta^{\alpha\beta} \\
J_2 &\equiv \frac{1}{2}\Big(J_1^2 - C_{\alpha\beta}C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big) \\
J_3 &\equiv \frac{1}{6}\Big(J_1^3 - 3J_1 C_{\alpha\beta}C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta} + 2C_{\alpha\beta}C_{\gamma\delta}C_{\tau\sigma}\delta^{\alpha\gamma}\delta^{\delta\sigma}\delta^{\beta\tau}\Big) \\
K_1 &\equiv 2Q_\beta N_\alpha\delta^{\alpha\beta} + N_\beta N_\alpha\delta^{\alpha\beta}Q^2 \\
K_2 &\equiv \Big(N_\alpha Q_\beta\delta^{\alpha\beta}\Big)^2 - \Big(N_\alpha N_\beta C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big)Q^2 - \Big(Q_\alpha Q_\beta\delta^{\alpha\beta}\Big)\Big(N_\gamma N_\delta\delta^{\gamma\delta}\Big) + \\
&- 2\Big(N_\alpha Q_\beta C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big) \\
K_3 &\equiv \Big(Q^2 N_\alpha N_\beta C_{\gamma\delta}C_{\tau\sigma}\delta^{\alpha\gamma}\delta^{\delta\tau}\delta^{\sigma\beta}\Big) + 2\Big(N_\alpha Q_\beta C_{\gamma\delta}C_{\tau\sigma}\delta^{\alpha\gamma}\delta^{\delta\tau}\delta^{\sigma\beta}\Big) + \\
&+ \Big(Q_\alpha Q_\beta C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big)\Big(N_\alpha N_\beta\delta^{\alpha\beta}\Big) + \Big(N_\alpha N_\beta C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\delta\beta}\Big)\Big(Q_\alpha Q_\beta\delta^{\alpha\beta}\Big) + \\
&- 2\Big(N_\alpha Q_\beta C_{\gamma\delta}\delta^{\alpha\gamma}\delta^{\beta\delta}\Big)\Big(N_\alpha Q_\beta\delta^{\alpha\beta}\Big)
\end{aligned}$$

and where $c_{\alpha\beta}$, N_α, K_1, K_2, K_3, stand for $c'_{\alpha\beta}$, N'_α, $'K_1$, $'K_2$, $'K_3$, if $I_A = I'_A$ and for $c''_{\alpha\beta}$, N''_α, $''K_1$, $''K_2$, $''K_3$, if $I_A = I''_A$, respectively. Together with the invariants J_1, J_2, J_3, $'K_1$, $'K_2$, $'K_3$, $''K_1$, $''K_2$, $''K_3$, we shall introduce the denotations

$$B^{\gamma\delta} \equiv \frac{\partial J_2}{\partial C_{\gamma\delta}} = C_{\alpha\beta}\Big(\delta^{\gamma\delta}\delta^{\alpha\beta} + \delta^{\gamma\alpha}\delta^{\delta\beta}\Big)$$

$$
\begin{aligned}
E^{\gamma\delta} &\equiv \frac{\partial K_3}{\partial C_{\gamma\delta}} = \Big(Q^2 N_\beta + 2Q_\beta\Big) N_\alpha C_{\tau\sigma} \delta^{\alpha\gamma}\delta^{\delta\tau}\delta^{\sigma\beta} + \\
&+ \Big(Q_\alpha Q_\beta \delta^{\alpha\gamma}\delta^{\beta\delta}\Big)\Big(N_\alpha N_\beta \delta^{\alpha\beta}\Big) + \Big(N_\alpha N_\beta \delta^{\alpha\gamma}\delta^{\beta\delta}\Big)\Big(Q_\alpha Q_\beta \delta^{\alpha\beta}\Big) + \\
&- 2\Big(N_\alpha Q_\beta \delta^{\alpha\gamma}\delta^{\beta\delta}\Big)\Big(N_\alpha Q_\beta \delta^{\alpha\beta}\Big) \\
D^{\gamma\delta} &\equiv \frac{\partial K_2}{\partial C_{\gamma\delta}} = -\Big(N_\beta Q^2 + 2Q_\beta\Big) N_\alpha \delta^{\alpha\gamma}\delta^{\beta\delta}
\end{aligned}
$$

and note that

$$
\frac{\partial J_1}{\partial C_{\gamma\delta}} = \delta^{\gamma\delta} \qquad \frac{\partial J_3}{\partial C_{\gamma\delta}} = J_3 C^{\gamma\delta} \qquad \frac{\partial K_1}{\partial C_{\gamma\delta}} = 0
$$

where, as before, $c_{\alpha\beta}$, N_α, K_1, K_2, K_3, stand for $c'_{\alpha\beta}$, N'_α, $'K_1$, $'K_2$, $'K_3$, if $I_A = I'_A$ and for $c''_{\alpha\beta}$, N''_α, $''K_1$, $''K_2$, $''K_3$, if $I_A = I''_A$, respectively. From Eqs (6.1) it follows that here and in the sequel(summation over $A = 1, 2, 3$ holds!)

$$
\begin{aligned}
\frac{\partial U'}{\partial C_{\gamma\delta}} &= \frac{\partial U'}{\partial I'_A}\frac{\partial I'_A}{\partial C_{\gamma\delta}} \\
\frac{\partial U''}{\partial C_{\gamma\delta}} &= \frac{\partial U''}{\partial I''_A}\frac{\partial I''_A}{\partial C_{\gamma\delta}}
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
\frac{\partial I'_1}{\partial C_{\gamma\delta}} &= \frac{\partial I'_1}{\partial c'_{\gamma\delta}} = \delta^{\gamma\delta} \\
\frac{\partial I'_2}{\partial C_{\gamma\delta}} &= \frac{\partial I'_2}{\partial c'_{\gamma\delta}} = {'D^{\gamma\delta}} + \delta^{\gamma\delta\prime} K_1 + B^{\gamma\delta} \\
\frac{\partial I'_3}{\partial C_{\gamma\delta}} &= \frac{\partial I'_3}{\partial c'_{\gamma\delta}} =' K_2 \delta^{\gamma\delta} +' K_1 B^{\gamma\delta} + C^{\gamma\delta} + J'_1 D^{\gamma\delta} +' E^{\gamma\delta}
\end{aligned}
\tag{6.3}
$$

and the similar formulae hold for I''_A and $c''_{\gamma\delta}$. Substituting Eqs (6.2) into Eqs (5.1) we obtain

$$
\begin{aligned}
S^{\alpha\beta} &= 2\rho\Big(\frac{\partial W'}{\partial I'_A}\frac{\partial I'_A}{\partial C_{\beta\alpha}}\sigma' + \frac{\partial W''}{\partial I''_A}\frac{\partial I''_A}{\partial C_{\beta\alpha}}\sigma''\Big) \\
H^{\alpha} &= 2\rho\Big(N'_\gamma\frac{\partial W'}{\partial I'_A}\frac{\partial I'_A}{\partial C_{\gamma\alpha}}\sigma' + N''_\gamma\frac{\partial W''}{\partial I''_A}\frac{\partial I''_A}{\partial C_{\gamma\alpha}}\sigma''\Big)
\end{aligned}
\tag{6.4}
$$

Eqs (6.4) combined with Eqs (6.3) represent the general form of constitutive equations for highly-elastic laminates made of the isotropic layers.

7. Incompressibility

Now assume that one from the isotropic constituents of the laminated body is made of an incompressible material. Under incompressibility condition $\det c'_{\alpha\beta} = 1$ leading to

$$\det\Big(C_{\alpha\beta} + 2N'_{(\alpha}Q_{\beta)} + N'_\alpha N'_\beta Q\Big) = 1 \tag{7.1}$$

we obtain $W'(I'_1, I'_2)$ and from Eqs (6.4) (subscripts A, B run over 1,2,3 and 1,2, respectively, here and in the sequel the summation convention holds for both A and B)

$$\begin{aligned} S^{\alpha\beta} &= 2\rho\Big(\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\beta\alpha}}\sigma' + \frac{\partial W''}{\partial I''_A}\frac{\partial I''_A}{\partial C_{\beta\alpha}}\sigma''\Big) + C^{\alpha\beta}p' \\ H^\alpha &= 2\rho\Big(N'_\gamma\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\gamma\alpha}}\sigma' + N''_\gamma\frac{\partial W''}{\partial I''_A}\frac{\partial I''_A}{\partial C_{\gamma\alpha}}\sigma''\Big) + C^{\alpha\gamma}N'_\gamma p' \end{aligned} \tag{7.2}$$

where p' represents what will be called the partial pressure in the first constituent of the laminate. If both constituents are isotropic and incompressible we obtain two incompressibility conditions

$$\begin{aligned} \det\Big(C_{\alpha\beta} + 2N'_{(\alpha}Q_{\beta)} + N'_\alpha N'_\beta Q\Big) &= 1 \\ \det\Big(C_{\alpha\beta} + 2N'_{(\alpha}Q_{\beta)} + N''_\alpha N''_\beta Q\Big) &= 1 \end{aligned} \tag{7.3}$$

and instead of Eqs (7.2) we shall write

$$\begin{aligned} S^{\alpha\beta} &= 2\rho\Big(\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\beta\alpha}}\sigma' + \frac{\partial W''}{\partial I''_B}\frac{\partial I''_B}{\partial C_{\beta\alpha}}\sigma''\Big) + C^{\alpha\beta}(p' + p'') \\ H^\alpha &= 2\rho\Big(N'_\gamma\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\gamma\alpha}}\sigma' + N''_\gamma\frac{\partial W''}{\partial I''_B}\frac{\partial I''_B}{\partial C_{\gamma\alpha}}\sigma''\Big) + C^{\alpha\gamma}\Big(N'_\gamma p' + N''_\gamma p''\Big) \end{aligned} \tag{7.4}$$

where p', p'' are partial pressures in both components.

Setting $I = J_3 = \det C_{\alpha\beta}$, we obtain

$$I_3 = \det c_{\alpha\beta} = I + K_3 + K_2 J_1 + K_1 J_2$$

and hence in Eqs (6.4) we can assume

$$\frac{\partial W'}{\partial I'_B} = \frac{\partial W'}{\partial I} \qquad\qquad \frac{\partial W''}{\partial I''_B} = \frac{\partial W''}{\partial I}$$

Let us introduce what will be called the macro-incompressibility condition

$$\det C_{\alpha\beta} = 1 \tag{7.5}$$

Then remembering that $N'_\gamma\sigma' + N''_\gamma\sigma'' = 0$, from Eqs (7.2) we derive the formulae

$$\begin{aligned} S^{\alpha\beta} &= 2\rho\Big(\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\beta\alpha}}\sigma' + \frac{\partial W''}{\partial I''_B}\frac{\partial I''_B}{\partial C_{\beta\alpha}}\sigma''\Big) + C^{\alpha\beta}p \\ H^\alpha &= 2\rho\Big(N'_\gamma\frac{\partial W'}{\partial I'_B}\frac{\partial I'_B}{\partial C_{\gamma\alpha}}\sigma' + N''_\gamma\frac{\partial W''}{\partial I''_B}\frac{\partial I''_B}{\partial C_{\gamma\alpha}}\sigma''\Big) \end{aligned} \tag{7.6}$$

where p will be called the macro-pressure. It can be seen that if the partial pressures p', p'' in Eqs (7.4) are interrelated by $p' = \sigma' p$, $p'' = \sigma'' p$, where p is the macro-pressure, then Eqs (7.4) imply Eqs (7.6).

For laminae made of the neo-hookean materials we have

$$W' = K'(I' - 3) \qquad\qquad W'' = K''(I'' - 3)$$

where K', K'' are material constants. In this case we obtain from Eqs (7.4) the explicit form of constitutive relations

$$\begin{aligned} S^{\alpha\beta} &= 2\rho C^{\alpha\beta}\Big(\sigma' K' + \sigma'' K''\Big) + C^{\alpha\beta}(p' + p'') \\ H^\alpha &= 2\rho C^{\alpha\beta}\Big(N'_\beta\sigma' K' + N''_\beta\sigma'' K''\Big) + C^{\alpha\gamma}\Big(N' p'_\gamma + N'' p''_\gamma\Big) \end{aligned}$$

which has to be considered together with the incompressibility conditions (7.3).

8. Conclusions

In this contribution it was shown that for the laminated structures, made of two isotropic highly-elastic constituents which are periodic in a certain reference configuration, it is possible to formulate the approximate governing relations which do not involve any macro-oscillating functions and hence can be applied both to the analysis and numerical calculations of engineering problems. The main results are given in the form of equations of motion (4.11) and the constitutive relations (6.4) and (7.2), (7.4), (7.6) for incompressible materials. The results can be easily generalized on the case of laminates made of a larger number of constituents; this generalization requires introduction of a large number of micro-shape functions and correctors. The characteristic feature of the proposed approach is the possibility of describing, on the macro-level, the micro-dynamical behaviour of the composites due to the presence of the micro-inertial terms in Eqs (4.2) and (4.11). That is why the proposed model can be referred to as the refined macrodynamics (using the terminology introduced by Woźniak (1993c)) or the macro-micro elastodynamics of highly-elastic laminates. The obtained equations of motion (4.2), (4.11) involve the microstructure length-dimension parameter l and hence describe the dispersion and scale-length effects due to the micro-inhomogeneity of a composite; such problems cannot be analyzed within a framework of the known homogenization approaches which are based on the asymptotic approximation $l \searrow 0$ (cf Turbe and Maugin (1991), Tartar (1990), Francfort and Marat (1992), Boutin and Auriault (1993), Cherkaev (1993), Wągrowska (1986) and (1988), Nagórko and Matysiak (1989), Nagórko (1989), Kaczyński and Matysiak (1988)). The applications of the derived equations to the analysis of special problems will be reserved for the subsequent investigations.

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Nieliniowa makro-mikro dynamika struktur laminowanych

Streszczenie

Celem opracowania jest znalezienie makro-modelu periodycznie laminowanego ośrodka zbudowanego z dwóch izotropowych sprężystych składników i poddanego skończonym odkształceniom. W proponowanym podejściu otrzymane równania zależą od charakterystycznego wymiaru mikrostruktury i dlatego opisują zarówno zjawiska dyspersyjne jak i swobodne drgania wysokoch częstotliwości dla rozważanych laminatów.

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