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ON THE MICRO-DYNAMICS OF COMPOSITE MATERIALS[1]

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In order to investigate the microstructure lenght-scale effect on a dynamic response of a composite body, a new general modelling approach to periodic material structures is proposed. Applications of the resulting equations to the micro-vibration analysis are illustrated by numerical examples. Considerations are restricted to the linear elastic composites with a perfect bonding between constituents.

1. Introduction

The existing macro-modelling methods for elastic periodic composites are mostly based on the homogenization approaches, leading to the concept of a homogeneous equivalent body. As it is known, material properties of this body are determined by so called effective modulus theories. The list of references to this subject is rather extensive, monographs by Bensoussan et al. (1978), Sanchez-Palencia (1980), Bakhavalov and Panasenko (1984), Aboudi (1991), Nemat-Nasser and Hori (1993), can be mentioned as showing the general lines of macro-modelling approach. However, the effective modulus theories are not able to describe the effect of the microstructure length dimensions on the macro-behaviour of the periodic composite medium. This effect plays an important role mainly in dynamic problems and has been analysed

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by means of modelling procedures developed separately for some special periodic structures (cf Aboudi (1981); Achenbach and Herrmann (1968); Green and Naghdi (1965); Grot and Achenbach (1970); Hegemier (1972); Sun et al. (1968); Tiersten and Jahanmir (1977); Tolf (1983); and others). In this contribution there is proposed a new general approach to the formulation of averaged models for elasto-dynamics of periodic composites, which takes into account the microstructure lenght scale effect on overall properties of the body. The main advantage of this approach is a relatively simple form of resulting equations which can be applied to the analysis of engineering problems and constitute the basis for numerical calculations. Moreover, for quasi-stationary problems the aforementioned equations describe a certain, special effective modulus theory, which was independently formulated and applied in a series of paper by Bielski and Matysiak (1992), Kaczyński (1993), (1994), Kaczyński and Matysiak (1989), Matysiak (1989), (1992), Matysiak and Woźniak (1987), Naniewicz (1987), Wągrowska (1988), Woźniak (1987). The new approach proposed in this contribution is a certain alternative to that leading to the equations of the refined macrodynamics, developed in a series of papers by Woźniak (1993), Mazur-Śniady (1993), Wierzbicki (1995), Mielczarek and Woźniak (1995), Jędrysiak and Woźniak (1995), Baron and Woźniak (1995), Matysiak and Nagórko (1995), Michalak et al. (1995), Cielecka (1995), Wągrowska and Woźniak (1995).

Notations. Tensorial subscripts $i, j, k, \ldots$ are related to the carthesian orthogonal coordinate system $0x_1x_2x_3$ in the physical three-space E and hence run over the sequence $1, 2, 3$. Non-tensorial superscripts $A, B, \ldots$ run over $1, \ldots, N$. For all the aforementioned indices summation convention holds unless otherwise stated. Points of the physical space are denoted by $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3)$, etc., and t stands for time coordinate; fourtuples (x_1, x_2, x_3, t) are assumed to represent inertial coordinates in the space-time. The representative valume element of a periodic material structure is denoted by $V = (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ and $l \equiv \sqrt{l_1^2 + l_2^2 + l_3^2}$ is said to be the microstructure length parameter. It is assumed that l is small enough as compared to the smallest characteristic length dimension L of the region Ω in E, occupied by the composite body in its natural state. An arbitrary translation of V by a radius vector $\boldsymbol{x}$ is defined by $V(\boldsymbol{x}) \equiv V + \boldsymbol{x}$, the region $\Omega^0 \equiv \{\boldsymbol{x} \in \Omega : \ V(\boldsymbol{x}) \subset \Omega\}$ is referred to as the macro-interior of Ω and $\Omega \setminus \Omega^0$ is called the near-boundary layer. For any integrable (time dependent) function $f(\cdot, t)$ defined on Ω, its averaged value over $V(x)$, $x \in \Omega^0$ is defined by

$$\langle f\rangle(x,t) \equiv \frac{1}{l_1 l_2 l_3} \int\limits_{V(x)} f(y,t)\, dv(y) \qquad dv(y) \equiv dy_1 dy_2 dy_3 \qquad x \in \Omega^0$$

If f is a time-independent V-periodic function, then

$$\langle f\rangle \equiv \frac{1}{l_1 l_2 l_3} \int\limits_{V(x)} f(y)\, dv(y)$$

represents, for any $x \in \Omega^0$, its averaged (constant) value. The components of displacements, strains, stresses and body forces will be denoted by u_i, ε_{ij}, σ_{ij}, b_i, respectively; for sake of simplicity the body forces are assumed to be constant.

2. Foundations

The subject of analysis is a heterogeneous linear-elastic body occupying in its natural state the region Ω of the physical space. The elastic modulus and mass density fields, denoted by $c_{ijkl}(\cdot)$ and $\rho(\cdot)$, respectively, will be treated as V-periodic functions. Restricting considerations to composite materials the above functions have to be assumed as piecewise constant, i.e., constant in regions occupied by each constituent. The main aim of analysis is to propose a new macro-modelling procedure which makes it possible to determine properties of this body in a certain averaged manner but depending explicity on the microstructure length parameter l. Hence the resulting macro-model has to describe the effect of size of the representative volume element V on the global behaviour of the composite.

The proposed approach is based on certain modelling hypotheses. In order to formulate those hypotheses, two auxiliary concepts have to be defined (cf Woźniak (1993)).

First, it will be assumed that the periodic heterogeneous structure of the solid under consideration implies certain disturbances in a displacement field. These disturbances from the qualitative wievpoint will be described by a certain postulated a priori system $h^A(\cdot)$, $A = 1,...,N$, of continuous piecewise differentiable V-periodic linear-independent real valued functions, satisfying conditions: $\langle h^A\rangle = 0$, $h^A(x) \in \mathcal{O}(l)$ and the extremum values of $h^A{}_{,\alpha}$ are independent of the microstructure length parameter l. Functions $h^A(\cdot)$ will be called micro-shape functions. As an example we can take

$h^A = l\sin(2n_A^1\pi x_1/l_1)\sin(2n_A^2\pi x_2/l_2)\sin(2n_A^3\pi x_3/l_3)$, where n_A^1, n_A^2, n_A^3 are arbitrary positive integers. The choice of micro-shape functions will determine the character (shape) of disturbances we are going to investigate in the problem considered.

Second, to every real valued function $F(x)$, $\boldsymbol{x} \in \Omega$, introduced as a certain kinematic variable, a numerical accuracy ε_F related to calculations of its values will be assigned. Function $F(\cdot)$ will be called a macro-function if for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ condition $\|\boldsymbol{x} - \boldsymbol{y}\| < l$ implies $|F(\boldsymbol{x}) - F(\boldsymbol{y})| < \varepsilon_F$. If $F(\cdot)$ is a differentiable function and similar conditions hold also for all derivatives of $F(\cdot)$ (with pertinent numerical accuracies related to those derivatives) then $F(\cdot)$ is said to be a regular macro-function. Hence increments of any macro-function within an arbitrary but fixed cell $V(x)$, $\boldsymbol{x} \in \Omega$, from a numerical viewpoint are small and can be neglected.

Using the aforementioned concepts three modelling hypotheses will be introduced.

- *Kinematic Hypothesis (KH).*

  The displacement fields $u_i(\cdot,t)$ in every problem under consideration will be assumed in the form

$$u_i(x,t) = U_i(x,t) + h^A(x)Q_i^A(x,t) \qquad \boldsymbol{x} \in \Omega \tag{2.1}$$

  where $h^A(\cdot)$, $A = 1,...,N$ is the postulated a priori micro-shape function system and $U_i(\cdot,t)$, $Q_i^A(x,t)$ are arbitrary regular macro-functions.

The physical sense of KH is strictly related to the meaning of a micro-shape function. Macro-functions $U_i(\cdot,t)$, $Q_i^A(x,t)$, represent new kinematic variables and will be called macro-displacements and internal macro-parameters, respectively. It is evident that Q_i^A describe, from the quantitative viewpoint, the micro-disturbances of displacements; it will be shown that these disturbances are caused by periodic material structure of the composite.

As it is known, within the framework of micromechanics the governing equations of a linear elastic body are given by

$$(c_{ijkl}u_{k,l})_{,j} - \rho\ddot{u}_i + \rho b_i = 0$$

Under the kinematic constraints introduced by Eq (2.1) the above equations are satisfied only in a certain averaged form. This statement will be represented by the following assumption.

- *Averaging Assumption (AA).*

  The following averaged form of equations of motion

$$
\begin{aligned}
&\langle (c_{ijkl}u_{k,l})_{,j} - \rho\ddot{u}_i + \rho b_i\rangle(x,t) = 0 \\
&\langle [(c_{ijkl}u_{k,l})_{,j} - \rho\ddot{u}_i + \rho b_i]h^A\rangle(x,t) = 0
\end{aligned}
\qquad x \in \Omega \tag{2.2}
$$

  is assumed to hold.

- *Macro-Modelling Approximation (MMA).*

  In the calculations of averages in Eqs (2.2) terms $\mathcal{O}(\varepsilon_F)$ will be neglected as compared to the values $F(x)$ of an arbitrary macro-function F. In the sequel F runs over macro-functions $U_i(\cdot,t)$, $Q_i^A(x,t)$ and all their derivatives.

The physical meaning of MMA is implied by the concept of macro-function. Since in calculations of averages $h^A Q^A_{i\ ,\alpha} \in \mathcal{O}(\varepsilon_Q) + \mathcal{O}(\varepsilon_{\nabla Q})$ we obtain

$$
(h^A Q_i^A)_{,\alpha} = h^A{}_{,\alpha} Q_i^A + \mathcal{O}(\varepsilon_Q) + \mathcal{O}(\varepsilon_{\nabla Q}) \tag{2.3}
$$

Similary, for an arbitrary integrable V-periodic function $f(\cdot)$ and any integrable macro-funtion $F(\cdot,t)$, we obtain

$$
\langle fF\rangle(x,t) = \langle f\rangle F(x,t) + \mathcal{O}(\varepsilon_F) \tag{2.4}
$$

By means of MMA terms $\mathcal{O}(\varepsilon_Q)$, $\mathcal{O}(\varepsilon_{\nabla Q})$ in Eqs (2.3), (2.4) will be neglected. It has to be emphasized that MMA has nothing in common with an asymptotic approximation procedure since terms $\mathcal{O}(\varepsilon_F)$ are neglected only as compared to the values $F(x)$ of an arbitrary macro-function $F(\cdot)$.

3. Results

Substituting the right-hand sides of Eqs (2.1) into Eqs (2.2), using MMA, i.e., neglecting terms $\mathcal{O}(\varepsilon_Q)$, $\mathcal{O}(\varepsilon_F)$ in formulae of the form (2.3), (2.4), and taking into account that $c_{ijkl}(\cdot)$, $\rho(\cdot)$ are V-periodic functions, after rather lenghty calculations not given here, we arrive at the system of equations in U_i and Q_i^A.

Let us introduce functions $k^A(\cdot)$ defined by $k^A \equiv h^A l^{-1}$; it can be seen that extremal values of $k^A(x)$ are independent of the microstructure length parameter l. Setting

$$\begin{aligned} S_{ij}(x,t) &= \langle c_{ijkl}\rangle U_{(k,l)}(x,t) + \langle c_{ijkl}h^A{}_{,l}\rangle Q_k^A(x,t) \\ H_i^A(x,t) &= \langle c_{ijkl}h^A{}_{,j}\rangle U_{(k,l)}(x,t) + \langle c_{ijkl}h^A{}_{,l}h^B{}_{,l}\rangle Q_k^B(x,t) \end{aligned} \tag{3.1}$$

the resulting averaged equations of motion have the form

$$\begin{aligned} S_{ij}(x,t) - \langle\rho\rangle\ddot{U}_i(x,t) - l\langle\rho k^A\rangle\ddot{Q}_i^A(x,t) + \langle\rho\rangle b_i = 0 \\ l^2\langle\rho k^A k^B\rangle\ddot{Q}_i^B(x,t) + l\langle\rho k^A\rangle\ddot{U}_i(x,t) + H_i^A(x,t) = 0 \end{aligned} \tag{3.2}$$

The above equations for every $x \in \Omega^0$ have been derived from Eqs (2.2). Since fields $S_{ij}(\cdot,t)$, $H_i^A(\cdot,t)$ are defined on Ω then Eqs (3.2) are well defined also in the near-boundary layer $\Omega \setminus \Omega^0$. Moreover, due to the condition $l \ll L$, formulated in Section 1, the layer $\Omega \setminus \Omega^0$ is relatively small. Hence in the sequel it will be assumed that Eqs (3.2) have to be satisfied for every $x \in \Omega$. It has to be remembered, however, that Eqs (3.2) have the physical interpretation as averages (2.2) only for $x \in \Omega^0$.

The obtained Eqs (3.1), (3.2) have constant coefficients and represent a certain macro-model of the periodic composite body. In the framework of this model material properties of the body are described by constant averaged modulae in Eqs (3.1); that is why these equations will be referred to as constitutive equations. The inertial properties are given by the averaged mass density $\langle\rho\rangle$ and the modulae $l\langle\rho k^A\rangle$, $l^2\langle\rho k^A k^B\rangle$ in equations of motion (3.2). Since in general $H_i^A(x,t) \in \mathcal{O}(l)$ and in the stationary processes $H_i^A = 0$, then $H_i^A(x,t)$ will be termed micro-dynamic forces. At same time $S_{ij}(x,t)$ are said to be macro-stresses. It can be shown that for every $x \in \Omega^0$ macro-stresses, micro-dynamic forces and macro-displacements have a simple physical interpretation given by

$$\begin{aligned} S_{ij}(x,t) &= \langle\sigma_{ij}\rangle(x,t) + \mathcal{O}(\varepsilon_{\nabla U}) + \mathcal{O}(\varepsilon_Q) \\ H_i^A(x,t) &= \langle\sigma_{ij}h^A{}_{,j}\rangle(x,t) + \mathcal{O}(\varepsilon_{\nabla U}) + \mathcal{O}(\varepsilon_Q) \qquad\qquad x \in \Omega^0 \\ U_i(x,t) &= \langle u_i\rangle(x,t) + \mathcal{O}(\varepsilon_Q) \end{aligned}$$

The above interpretation does not hold in the vicinity $\Omega \setminus \Omega^0$ of the boundary $\partial\Omega$. Hence boundary values of U_i and S_{ij} will be treated similarly as displacements and stresses in boundary conditions of solid mechanics.

Let us observe that all averaged modulae in Eqs (3.1), (3.2) are invariant under arbitrary rescalling of the representative volume element V. It follows that Eqs (3.2) depend explicitly on the microstructure length parameter l. Hence the proposed model describes in the explicit form the effect of the microstructure size on the global behaviour of the body. It is easy to conclude that this effect is caused by the inertia forces because in quasi-stationary processes all terms in Eqs (3.2) involving length parameter l are neglected.

Substituting the right-hand sides of Eqs (3.1) into Eqs (3.2) we arrive at the system $3(N+1)$ equations in macrodisplacements U_i and internal macro-parameters Q_i^A. The second characteristic feature of the model is that unknowns Q_i^A are governed by a system of ordinary differential equations involving exclusively time derivatives. That is why the macro-fields Q_i^A were referred to as the internal macro-parameters, being independent of the boundary conditions. This fact plays an important role in a formulation of initial-boundary value problems for Eqs (3.1), (3.2), where the boundary conditions can be postulated in a form similar to that known in the linear elasticity theory.

For a homogeneous body the conditions $\langle \rho k^A \rangle = \rho \langle k^A \rangle = 0$, $\langle c_{ijkl} h^A{}_{,j} \rangle = c_{ijkl} \langle h^A{}_{,j} \rangle = 0$ hold. In this case from Eqs (3.1), (3.2) we obtain the well known equations of the linear elasticity theory. Moreover, under homogeneous initial conditions all internal macro-parameters Q_i^A are equal to zero. Using Eqs (2.1) we jump to the conclusion that macro-parameters Q_i^A describe the disturbances in displacements caused by the inhomogeneity of the medium and/or by the initial distribution of these disturbances.

The analytical form of the proposed macro-models, for every special problem, depends on the choice of micro-shape functions. Hence in every problem under consideration a special class of micro-disturbances is investigates. It has to be emphasized that solutions to the problem considered have a physical sense only if U_i, Q_i^A are sufficiently regular macro-functions.

4. Stationary problems

For stationary problems the second one from Eqs (3.2) reduces to the form $H_i^A = 0$. Moreover, it can be also shown that the linear transformation $R^{3N} \rightarrow R^{3N}$ represented by $3N \times 3N$ matrix of elements $\langle c_{ijkl} h^A{}_{,j} h^B{}_{,l} \rangle$ is invertible. It follows that for stationary problems macro-parameters Q_i^A can be eliminated from governing equations by means of

$$Q_i^A(x) = -D_{ij}^{AB}\langle c_{jklm}h^B{}_{,k}\rangle U_{(l,m)}(x) \tag{4.1}$$

where terms D_{ij}^{AB} represent the pertinent inverse transformation

$$D_{ij}^{AB}\langle c_{jklm}h^B{}_{,k}h^C{}_{,l}\rangle = \delta^{AC}\delta_{im}$$

Denoting

$$C_{ijkl} \equiv \langle c_{ijkl}\rangle - \langle c_{ijmn}h^A{}_{,m}\rangle D_{np}^{AB}\langle c_{klpr}h^B{}_{,r}\rangle \tag{4.2}$$

we obtain the following equations describing stationary processes within the framework of the proposed macro-model of a composite body

$$\begin{aligned} S_{ij}(x) &= C_{ijkl}U_{(k,l)}(x) \\ S_{ij,j}(x) + \langle\rho\rangle b_i &= 0 \end{aligned} \tag{4.3}$$

From the formal viewpoint Eqs (4.3) have the form similar to that appearing in the linear elasticity theory for stationary problems. Terms C_{ijkl} defined by Eq (4.2) will be called the effective module of the composite material for deformations of the form (2.1). Hence for stationary problems the proposed model of a composite body reduces to a certain effective modulus theory. This theory has been applied to the analysis of engineering problems in a series of papers by Kaczyński (1994), Kaczyński and Matysiak (1991), Matysiak (1992), Naniewicz (1989), Wągrowska (1988), Wierzbicki (1989), Woźniak (1987).

5. Applications to micro-dynamics

In this section we shall apply the obtained general results to a special case of the micro-vibration analysis for composite materials under consideration. To this end assume that the micro-shape functions satisfy the extra condition $\langle\rho h^A\rangle = 0$, $A = 1,...,N$. Under this condition Eqs (3.1), (3.2) yield the following system of equations for U_i, Q_i^A

$$\begin{aligned} &\langle c_{ijkl}\rangle U_{k,lj} + \langle c_{ijkl}h^A{}_{,l}\rangle Q_{k,j}^A - \langle\rho\rangle\ddot{U}_i + \langle\rho\rangle b_i = 0 \\ &l^2\langle\rho k^A k^B\rangle\ddot{Q}_i^B + \langle c_{ijkl}h^A{}_{,j}h^B{}_{,l}\rangle Q_k^B + \langle c_{ijkl}h^A{}_{,j}\rangle U_{k,l} = 0 \end{aligned} \tag{5.1}$$

The above equations have been derived independently by Woźniak (1993). In the sequel we shall restrict consideration to a special class of solution to Eqs (5.1) given by

$$U_i = U_i(x) \qquad Q_i^A(x) = -D_{ij}^{AB}\langle c_{jklm}h^B{}_{,k}\rangle U_{(l,m)}(x) + V_i^A(t) \tag{5.2}$$

where macrodisplacements $U_i(x)$ satisfy Eqs (4.3) and regular macro-functions $V_i^A(t)$ are solutions to the system of $3N$ ordinary differential equations

$$l^2\langle\rho k^A k^B\rangle\ddot{V}_i^B + \langle c_{ijkl}h^A{}_{,j}h^B{}_{,l}\rangle V_k^B = 0 \tag{5.3}$$

At the same time using Eqs (2.1) and (5.2) we obtain

$$u_i(x,t) = U_i(x) - h^A(x)D_{ij}^{AB}\langle c_{jklm}h^B{}_{,k}\rangle U_{(l,m)}(x) + h^A(x)V_i^A(t) \tag{5.4}$$

Thus, we conclude that under extra conditions $\langle\rho k^A\rangle = 0$ imposed on k^A there exist the decomposition of the problem into stationary deformations described by the macro-displacement field U_i governed by Eqs (4.3) and superimposed micro-displacements $v_i^A(x,t) \equiv h^A(x)V_i^A(t)$ determined by the internal macro-parameters V_i^A satisfying Eqs (5.3). Since the form of Eqs (4.3) in macrodisplacements coincides with that of equations for homogeneous linear-elastic bodies, then the subsequent analysis will be restricted to Eqs (5.3). In order to simplify the considerations we shall restrict ourselves to the unbounded medium. Setting $V_i^B(t) = A_i^B\cos(\omega t)$, where A_i^B are arbitrary constants and ω is the micro-oscillation frequency, the non-trivial solution to Eqs (5.3) exists only if the well known frequency equation holds

$$\det\Big(\langle c_{ijkl}h^A{}_{,j}h^B{}_{,l}\rangle - \omega^2\langle\rho h^A h^B\rangle\delta_{ik}\Big) = 0 \tag{5.5}$$

The general analysis of conditions having the above form is well known and we shall pass to the investigations of special problems. For sake of simplicity let us confine ourselves to the one-dimensional space-problems, in which microdisplacements $v_i(\cdot,t)$ depend only on one spatial coordinate x_1. Let us assume that $v_i(x_1,t) = v_i(x_1+L_n,t)$ for every $x_1 \in R$, where the positive constant L_n represents the micro-oscilation wavelength. Due to the l_1-periodicity of micro-shape functions $h^A(x_1)$, $x_1 \in R$, only the wavelengths given by $L_n = l_1/n$, $n = 1,2,3,...$ can be taken into account. Thus the wavenumbers, defined by $K_n \equiv 2\pi/L_n$, belong to $\{2\pi n/l_1;\ n = 1,2,3,...\}$. For the above values of K_n functions $h^A(\cdot)$ will be assumed in the form $h^A(x_1) = \frac{1}{K_n}\cos(AK_nx_1)$, $A = 1,...,N$ for any fixed n. Let $x_1 = \mathrm{const}$ be the elastic symmetry planes for every constituent of the composite. In this case, under denotation (here and in the sequel no summation over the subscript i!)

$$
\begin{aligned}
C_i^{AB}(K_n) &\equiv AB\langle c_{i1i1}\sin(AK_nx_1)\sin(BK_nx_1)\rangle \\
\rho^{AB}(K_n) &\equiv \langle \rho\cos(AK_nx_1)\cos(BK_nx_1)\rangle
\end{aligned}
\tag{5.6}
$$

Eqs (5.5) yield the interrelation between ω and K_n

$$
\det\Big[C_i^{AB}(K_n) - \Big(\frac{\omega}{K_n}\Big)^2\rho^{AB}(K_n)\Big] = 0
\tag{5.7}
$$

for every wavenumber K_n. If $i = 1$ then Eqs (5.7) is the dispersion relation for the longitudinal waves, if $i = 2,3$ then it represents pertinent relations for the transversal waves. For a two-constituent composite medium, where there is a decomposition of V into two disjointed parts V', V'' occupied by different constituents, $\overline{V} = \overline{V'} \cup \overline{V''}$, $V' \cap V'' = 0$, for every $\boldsymbol{x} \in V' \cup V''$ we obtain

$$
c_{i1i1} = \begin{cases} c'_{i1i1} & \text{if} \quad \boldsymbol{x} \in V' \\ c''_{i1i1} & \text{if} \quad \boldsymbol{x} \in V'' \end{cases}
\qquad\qquad
\rho(\boldsymbol{x}) = \begin{cases} \rho' & \text{if} \quad \boldsymbol{x} \in V' \\ \rho'' & \text{if} \quad \boldsymbol{x} \in V'' \end{cases}
\tag{5.8}
$$

Denoting $[c_{i1i1}] = c''_{i1i1} - c'_{i1i1}$, $[\rho] = \rho'' - \rho'$, for every $A \neq B$

$$
\begin{aligned}
C_i^{AB}(K_n) &= \frac{[c_{i1i1}]AB}{l_1l_2l_3}\int\limits_{V''}\sin(AK_nx_1)\sin(BK_nx_1)\;dv(\boldsymbol{x}) \\
\rho^{AB}(K_n) &= \frac{[\rho]}{l_1l_2l_3}\int\limits_{V''}\cos(AK_nx_1)\cos(BK_nx_1)\;dv(\boldsymbol{x})
\end{aligned}
$$

Thus we conclude that all non-diagonal elements of $N \times N$ matrices of elements $C_i^{AB}(K_n) - (\omega/K_n)^2\rho^{AB}(K_n)$ in Eq (5.7) are linear functions of jumps $[c_{i1i1}]$, $[\rho]$ of material properties of the composite. For homogeneous materials $[c_{i1i1}] = 0$, $[\rho] = 0$ and for every mode N we obtain from Eq (5.7) the known result $(\omega/K_n)^2 = c_{i1i1}/\rho$. For heterogeneous periodic materials, with increasing values of N we formulate more detailed description of the micro-vibration problem in the framework of one-dimensional model. It has to be emphasized that this model can be applied solely to problems for which the oscillations of micro-displacements $v_i(\cdot,t)$ in directions of x_2- and x_3-axes are sufficiently small and can be neglected.

The numerical calculations related to the longitudinal vibrations and based on the formulas (5.6) $\div$ (5.8), were carried out for fibre-reinforced composites

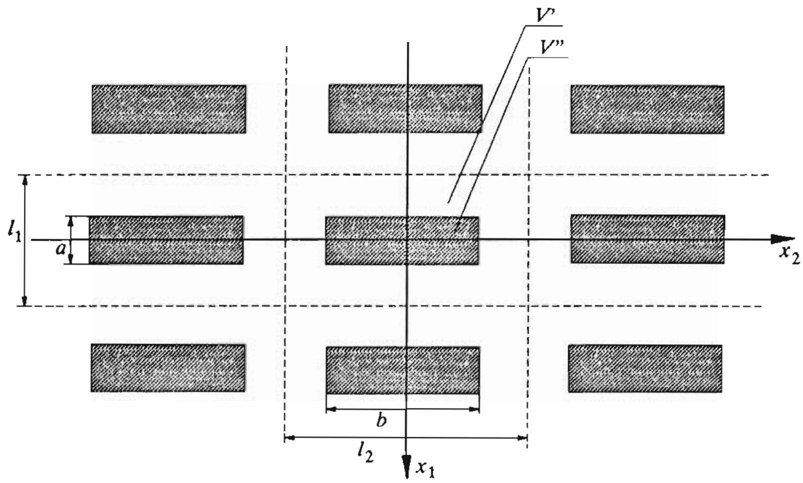

Fig. 1. Scheme of the slab – reinforced composite

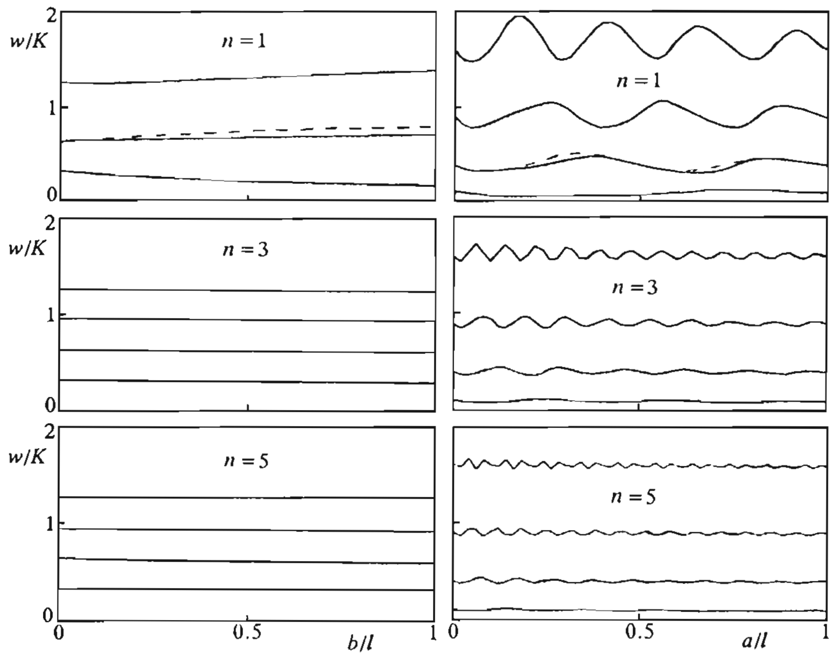

Fig. 2. ω/K_n versus b or a

the cross sections of which are shown in Fig.1, where $l = l_1 = l_2$, $\rho''/\rho' = 10$, $c''_{1111}/c'_{1111} = 10$ and $n = 1,3,5$ are taken as parameters. The results related to the cases $a = l/3$, $b \in [0,l]$ and $b = l/3$, $a \in [0,l]$ are given by diagrams in Fig.2 on the left and right-hand sides, respectively. The dashed lines are related to $N = 2$ and the continuous lines to $N = 4$. The diagrams of ω/K_n for $a \in [0,l], b \in [0,l]$ are shown in Fig.3 $\div$ Fig.5 for $n = 1,3,5$, respectively. The comparison of results obtained for $N = 2$ and $N = 4$ shows that the results obtained nearly coincide.

6. Conclusions

At the end of the paper let us summarize the main advantages and drawbacks of the proposed macro-modelling methods.

First, the obtained macro-models of periodic composite materials describe the microstructure length-scale effect on the behaviour of the body. This effect plays an important role in the analysis of dynamic problems, being neglected in the known homogenization theories which are restricted mainly to quasi-stationary problems.

Second, the proposed method is rather general, i.e., it can be applied to an arbitrary material structure of the representative element. For some special composite materials, such as laminates, the results derived from Eqs (3.1), (3.2) have been compared with those of the linear elasticity theory and the known effective stiffness theories leading to a satisfying approximation (cf Matysiak and Nagórko (1995)).

Third, the internal parameters Q_i^A in Eqs (3.1), (3.2) are governed by a system of ordinary differential equations and hence do not enter the boundary conditions. This fact plays an essential role in applications of the theory to engineering problems. At the same time the obtained governing equations have a simple analytical form and can be formulated by simple calculations of averaged modulae. It means that the proposed macro-modelling approach does not require any solution to a boundary-value problem on a unit cell, which is necessary in asymptotic homogenization methods.

On the other hand, Eqs (3.1), (3.2) can be formulated only if a certain system of macro-shape functions h^A is previously postulated. The choice of those functions depends on the character of the problem under consideration and in many special cases has to be based on the intuition of researcher. It means that the form of expected disturbances in the displacement field should be a priori postulated.

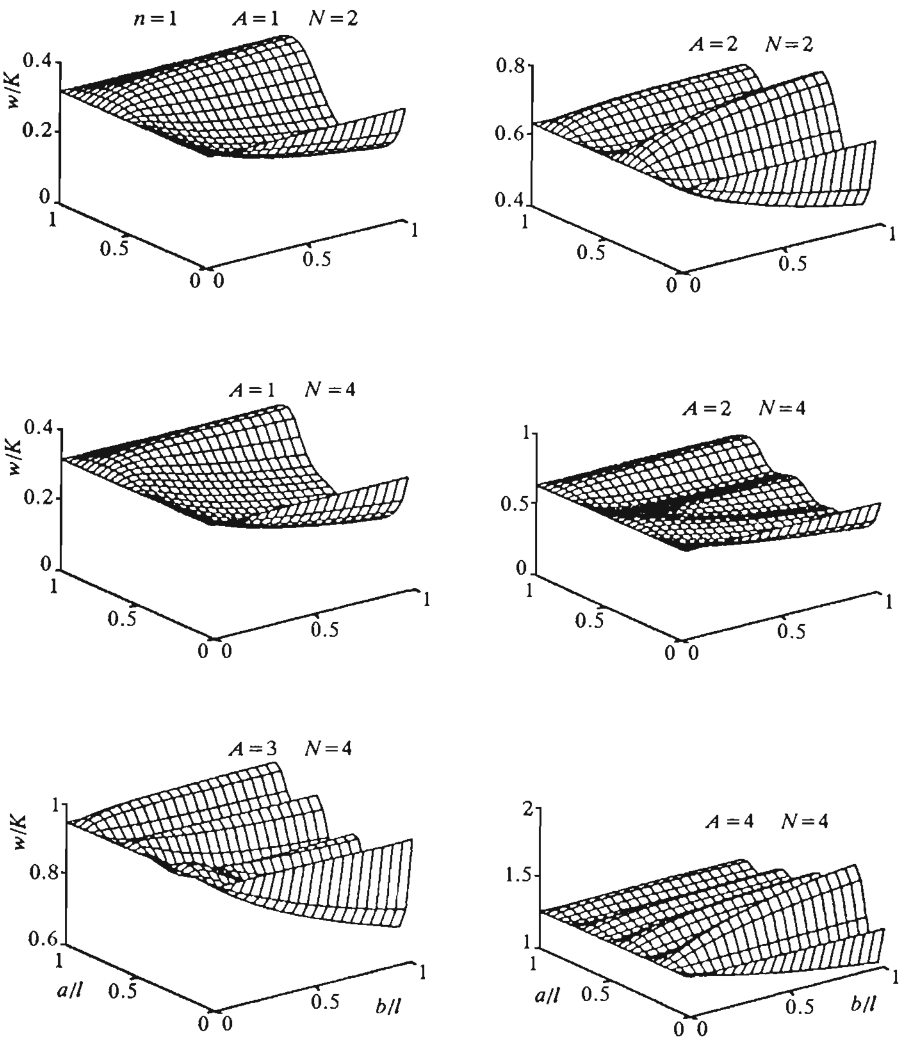

Fig. 3. A mesh surface ω/K_n above a rectangular grid in the $a-b$ plane for $n=1$

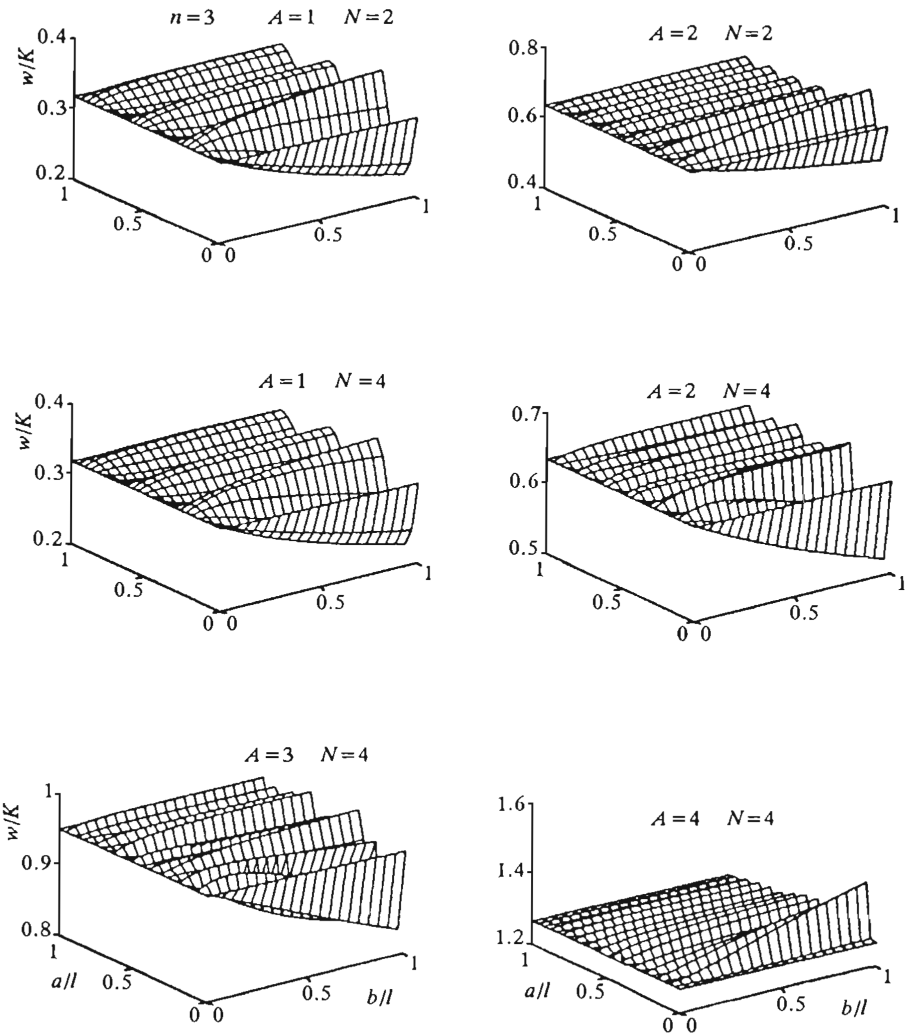

Fig. 4. A mesh surface ω/K_n above a rectangular grid in the $a-b$ plane for $n=3$

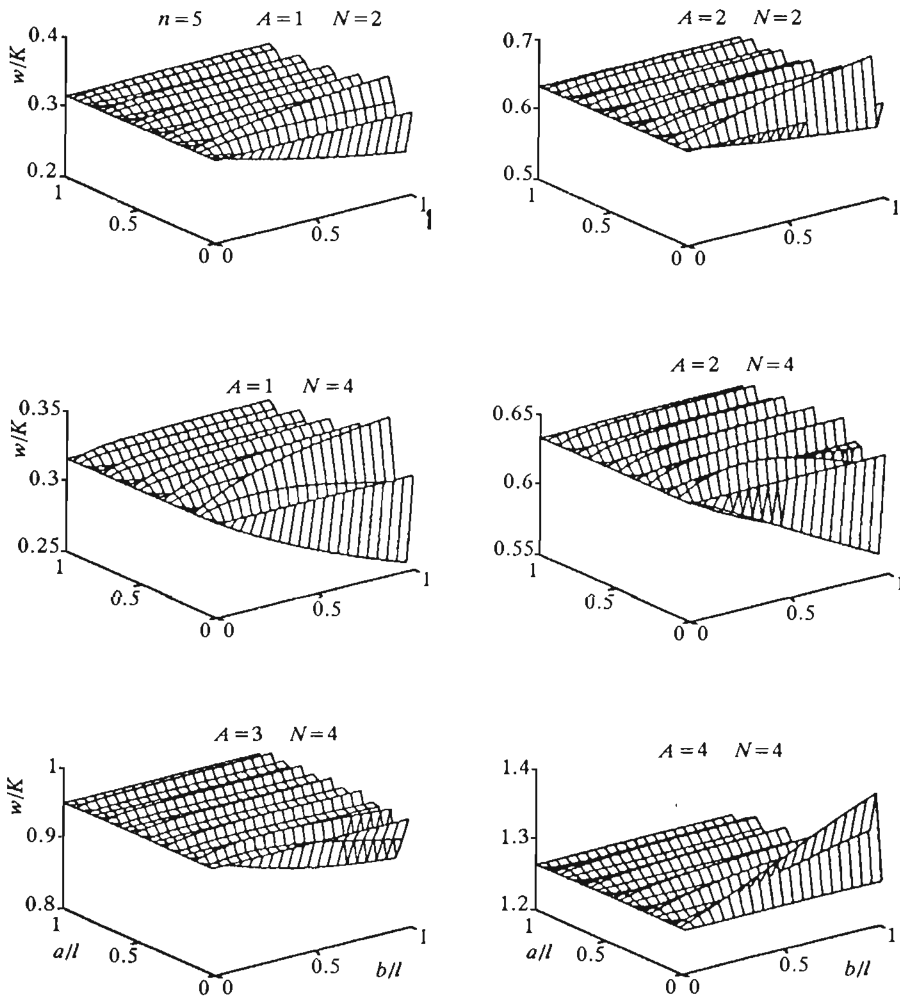

Fig. 5. A mesh surface ω/K_n above a rectangular grid in the $a-b$ plane for $n=5$

Applications of the theory have been illustrated by the simple example of micro-vibrations in a fibrous medium. For other applications the reader is referred to the papers mentioned in Section 1.

References

1. ABOUDI J., 1991, *Mechanics of Composite Materials: a Unified Micromechanical Approach,* Elsevier, Amsterdam
2. ACHENBACH J.D., HERRMANN G., 1968, Wave Motions in Solids with Lamellar Structuring, in *Dynamics of Structural Solids,* (edit. G.Herrmann), Am. Soc. Mech. Eng., New York
3. BAKHAVALOV N.S., PANASENKO G.P., 1984, *Averaged Processes in Periodic Media,* (in Russian), Nauka, Moscow
4. BARON E., WOŹNIAK C., 1995, On the Micro-Dynamics of Composite Plates, *Arch. Appl. Mech.,* 65 (in press)
5. BENSOUSSAN A., LIONS J.L., PAPANICOLAU G., 1978, *Asymptotic Analysis for Periodic Structures,* North-Holland, Amsterdam
6. BIELSKI W., MATYSIAK S.J., 1992, Surface Waves in a Periodic Two-Layered Elastic Half-Space, *Acta Mechanica,* **91**, 45-55
7. CIELECKA I., 1995, On the Continuum Modelling the Dynamic Behaviour of Certain Composite Lattice-Type Structures, *J. Theor. Appl. Mech.,* **33**, 351-360
8. GREEN A.E., NAGHDI P.M., 1965, A Dynamical Theory of Interacting Continua, *Int. J. Engng Sci.,* **3**, 231-241
9. GROT R.A., ACHENBACH J.D., 1970, Large Deformations of a Laminated Composite, *Int. J. Solids Structures,* **5**, 641-661
10. HEGEMIER G.A., 1972, On a Theory of Interacting Continua for Wave Propagation in Composites, in: *Dynamics of Composite Materials,* (edit. E.H.Lee), Am. Soc. Mech. Eng., New York
11. JĘDRYSIAK J., WOŹNIAK C., 1995, On the Elastodynamics of Thin Microperiodic Plates, *J. Theor. Appl. Mech.,* **33**, 337-350
12. KACZYŃSKI A., 1993, On the Three-Dimensional Interface Crack Problem in a Periodic Two-Layered Composites, *Int. J. Fracture,* **62**, 283-306
13. KACZYŃSKI A., 1994, Three-Dimensional Thermoelastic Problem of Interface Cracks in Periodic Two-Layered Composites, *Eng. Fract. Mech.,* **48**, 783-800
14. KACZYŃSKI A., MATYSIAK S.J., 1991, On the Stress Distribution in a Periodic Reinforced Elastic Composite with Partially Unbounded Fiber, *Eng. Fract. Mech.,* **39**, 745-756

15. MATYSIAK S.J., 1992, On the Homogenized Model of Periodic Stratified Fluid-Saturated Porous Solids, *Int. J. Engng Sci.*, **30**, 729-737

16. MATYSIAK S. J., NAGÓRKO W., 1995, On the Wave Propagation in Periodically Laminated Composites, *Bull. Pol. Ac. Sci., Sci. Techn.*, **43** (in press)

17. MAZUR-ŚNIADY K., 1993, Macro-Dynamics of Micro-Periodic Elastic Beams, *J. Theor. Appl. Mech.*, **31**, 781-793

18. MICHALAK B., WOŹNIAK C., WOŹNIAK M., 1995, The Dynamic Modelling of Elastic Wavy-Plates, *Arch. Appl. Mech.*, **65** (in press)

19. MIELCZAREK G., WOŹNIAK C., 1995, On the Dynamic Modelling of Fibrous Composites, *J. Tech. Phys.*, **36**, 103-111

20. NANIEWICZ Z., 1987, On the Homogenized Elasticity with Microlocal Parameters, *Bull. Acad. Polon. Sci., Sci. Techn.*, **35**, 399-409

21. NEMAT-NASSER S., HORI M., 1993, *Microdynamics: overall Properties of Heterogeneous Materials,* North-Holland, Amsterdam

22. SANCHEZ-PALENCIA E., 1980, Non-Homogeneous Media and Vibration Theory, *Lecture Notes in Physics,* 127, Springer, Berlin-Heidelberg-New York

23. SUN C.T., ACHENBACH J.D., HERRMANN G., 1968, Time-Harmonic Waves in a Stratified Medium Propagating in the Direction of Layering, *J. Appl. Mech.*, **35**, 408-411

24. TIERSTEN H.F., JAHANMIR M., 1977, The Theory of Composites Modelled as Interpenetrating Solid Continua, *Arch. Rat. Mech. Anal.*, **65**, 153-192

25. TOLF G., 1983, On Dynamical Description of Fibre-Reinforced Composites, in *New Problems in Mechanics of Continua,* (edit. Brulin and Hsieh), Study No 17, Waterloo Press, 145-159

26. WĄGROWSKA M., 1988, On the Homogenization of Elastic-Plastic Periodic Composites by Microlocal Parameter Approach., *Acta Mech.*, **67**, 45-65

27. WĄGROWSKA M., WOŹNIAK C., 1995, Macro-Modelling of Dynamic Problems for Visco-Elastic Composite Materials, *Int. J. Engng Sci.*, **33** (in press)

28. WIERZBICKI E., 1989, On the Effective Modulae in the Microlocal Models of Elastic Periodic Composites, *J. Theor. Appl. Mech.*, **27**, 33-40

29. WIERZBICKI E., 1995, Nonlinear Macro-micro Dynamics of Laminated Structures, *J. Theor. Appl. Mech.*, **33**, 1-17

30. WOŹNIAK C., 1987, A Nonstandard Method of Modelling of Thermoelastic Periodic Composites, *Int. J. Engng Sci.*, **25**, 133-141

31. WOŹNIAK C., 1993, Refined Macrodynamics of Periodic Structures, *Arch. Mech.*, **45**, 295-304

O mikrodynamice materiałów kompozytowych

Streszczenie

Praca omawia nowe podejście do modelowania zagadnień dynamiki periodycznych materiałów kompozytowych, uwzględniające wpływ wielkości mikrostruktury na globalne własności kompozytu. Otrzymane równanie modelu zastosowano do analizy pewnej klasy mikro-drgań ilustrując analizę przykładami numerycznymi.

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