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WEIGHT FUNCTIONS OF LOADING MODES I, II AND III FOR A ROUND HOLE WITH TWO SYMMETRIC RADIAL CRACKS

KRZYSZTOF L. MOLSKI

Institute of Cranes and Heavy Machinery Engineering
Warsaw University of Technology

Weight functions for a round hole with two symmetric radial cracks are derived for loading modes I and II using the boundary element method (BEM) together with the Bueckner type singular complex stress function at the crack tip. For the mode III – the asymptotic interpolation method has been employed. These three weight functions are then formulated in terms of correction and unitary weight functions and described in the unified form, suitable for computing stress intensity factors K. To assess the accuracy of present approach, the calculated values of K_1, K_2 and K_3 for various loading conditions are compared to the ones known from the literature or obtained from boundary element analysis.

Key words: crack, stress intensity factor, weight function

1. Introduction

A weight function method, proposed by Bueckner (1973) and Rice (1972), is one of the most efective tools in determination of unknown values of the stress intensity factors for different geometrical and loading conditions of the cracked body. The principle of superposition, which enables one to put together different linear elastic stress fields resulting from external loads, temperature gradients, residual stresses, etc., respectively, makes this approach very versatile and effective. If the resultant stresses along the crack sides are known, the corresponding stress intensity factors K_1, K_2 and K_3 are obtained by the following, simple integration

$$K_j = \int\limits_0^a \sigma_{1j}(x) m^{(j)}(x,a)\, dx \tag{1.1}$$

where

a	–	crack length
$m^{(j)}(x,a)$	–	known weight functions suitable for the cracked body
$\sigma_{1j}(x)$	–	components of the stress tensor released on the crack surface in the directions corresponding to the three loading modes $j = 1,2,3$.

Some improvements in the method can be found in Molski (1992) and (1994), where a unitary weight function (UWF) was defined. Its unified description and simplified integrating procedure appeared to be very effective tool for quick K determination (cf Molski and Truszkowski (1995)).

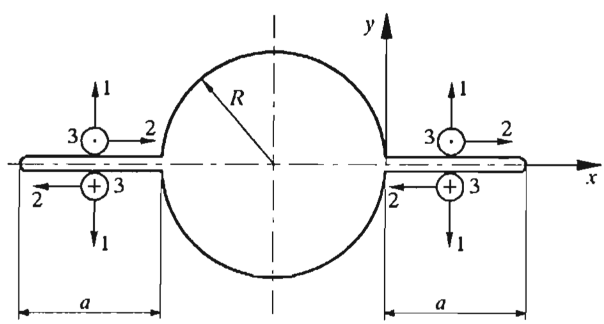

Fig. 1. Symmetric case of a circular hole with two radial cracks

In the present study a round hole with two equal and opposite radial cracks loaded symmetrically with respect to both axes of the hole, as shown in Fig.1, have been analysed. The normal and tangent stress components $\sigma_{1j}(x)$ are released on the crack faces and result in the multiaxial stress state, represented by the three basic modes I, II and III, and described at the vicinity of the crack tip by three different stress intensity factors K_1, K_2 and K_3. The aim of the present work is to determine weight functions corresponding to particular modes.

2. Determination of weight functions

2.1. Method for modes I and II

The particular values of the weight functions for modes I and II have been obtained using the boundary element technique (cf Portela and Aliabadi

(1993)) together with the complex stress function $Z_j(z)$ true in the vicinity of the crack tip

$$Z_j(z) = \frac{B_j}{\sqrt{z^3}} \tag{2.1}$$

where $z = re^{i\varphi}$ is a complex number in the polar coordinate system originated at the crack tip and $B_j = P_j\sqrt{c}/\pi$ – Bueckner's parameter, which represents the strength of singularity of the stress field for modes I $(j = 1)$ and II $(j = 2)$, respectively, for a pair of self-equilibrated forces P_j applied at a small distance c from the crack tip.

According to the method suggested by Bueckner (1973) and Tada et al. (1973), if a known stress field is applied to the crack tip under consideration, displacements of the remaining parts of the element represent the weight function components. However the method provides weight functions valid for the whole body, only the solutions of $m^{(j)}(x,a)$ related to the crack line are of interest here.

In order to improve accuracy of the numerical analysis, a small semicircle with its diameter equal to 0.5% of the total crack length has been built in the crack tip by 8 circular boundary elements, while the remaining part of the body, including one crack face, is mapped by 175 circular and straight boundary elements. Due to double symmetry of the problem, only one quarter of the hole and one crack face have been modelled. The Bueckner-type plane strain displacement fields u_1, v_1 and u_2, v_2 given by Eqs (2.2) and (2.3) and corresponding to B_1 and B_2, respectively, (cf Tada et al. (1973)), have been converted into boundary tractions and imposed at 17 nodes of the semicircle, as the boundary conditions of the problem, since the remaining part of the cracked body is traction-free

$$\begin{aligned} u_1 &= \frac{2(1+\nu)B_1}{E\sqrt{r}}\cos\frac{\varphi}{2}\Big(2\nu - 1 + \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}\Big) \\ v_1 &= \frac{2(1+\nu)B_1}{E\sqrt{r}}\sin\frac{\varphi}{2}\Big(2 - 2\nu - \cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}\Big) \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_2 &= \frac{2(1+\nu)B_1}{E\sqrt{r}}\sin\frac{\varphi}{2}\Big(2 - 2\nu + \cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}\Big) \\ v_2 &= \frac{2(1+\nu)B_1}{E\sqrt{r}}\cos\frac{\varphi}{2}\Big(1 - 2\nu + \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}\Big) \end{aligned} \tag{2.3}$$

Thus, the calculated crack face displacements – opening for mode I and sliding for mode II – have been analysed separately and interpreted as the displacement weight functions.

Since the weight functions in this case do not depend on the elastic material constants E and ν, their values have been conveniently chosen as $E = 1$ and $\nu = 0$ to simplify the output data analysis.

2.2. Correction and unitary weight functions

The next step consists in transforming numerically obtained weight functions for two loading modes I and II, into correction and unitary weight functions (cf Molski (1992), (1994)). To facilitate the description of the weight functions in the whole domain, a new parameter $s = a/(a+R)$ has been introduced, where $s \in < 0, 1 >$.

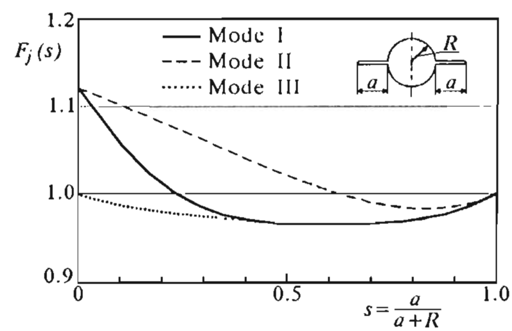

Fig. 2. Correction functions of modes I, II and III vs. s parameter

Two different correction functions: $F_1(s)$ and $F_2(s)$ shown by the solid and dashed lines in Fig.2, have been obtained by numerical integration of the previously found and normalized weight displacement functions. They depend only on the value of parameter s and express the influence of uniform and symmetric normal σ_{11} and tangential σ_{12} stresses, respectively, applied directly to the crack surfaces, on K_1 and K_2. Numerical values of the correction functions $F_1(s)$ and $F_2(s)$ interpolated by polynomials are given by the following equations

$$F_1(s) = 1.1215 - 0.827s + 1.627s^2 - 1.719s^3 + 1.788s^4 - 1.9035s^5 + 0.913s^6$$
$$F_2(s) = 1.1215 - 0.197s - 0.112s^3 + 0.1875s^4 \tag{2.4}$$

The values of K_1 and K_2 can be written as follows

$$K_1 = \sqrt{\pi a}\sigma_{11}F_1(s) \qquad K_2 = \sqrt{\pi a}\sigma_{12}F_2(s) \tag{2.5}$$

According to the procedure described by Molski (1992) and (1994), any form of classical weight function can be normalized and transformed into the unitary weight function $w(s)$, integral of which along any crack length always equals one.

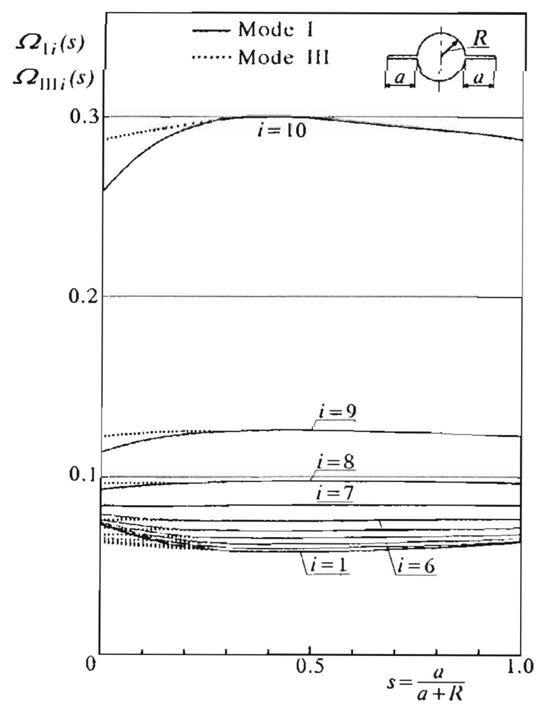

Fig. 3. Fractional values of the unitary weight function integrals (weight coefficients) for modes I and III – $\Omega_{Ii}(s)$ and $\Omega_{IIIi}(s)$ vs. s parameter

For numerical purposes in the case of non-uniform stress along the crack path, it is more convenient to use the fractional values of the unitary weight function integrals $\Omega_i(s)$, which are obtained by dividing the whole crack length a into ten equal segments i and integrating the $w(s)$ function numerically, starting from the crack end opposite to the considered crack tip. The fractional integral values $\Omega_{Ii}(s)$ and $\Omega_{IIi}(s)$ of the unitary weight functions vs. the shape parameter s are shown in Fig.3 and Fig.4. They are interpreted as the weight coefficients, indicating the influence of each tenth of the crack on the corresponding stress intensity factor value.

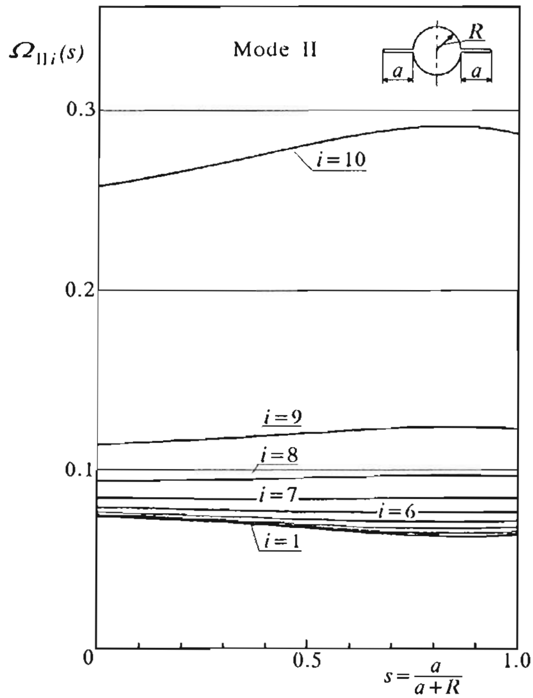

Fig. 4. Fractional values of the unitary weight function integrals (weight coefficients) for the mode II – $\Omega_{IIi}(s)$ vs. s parameter

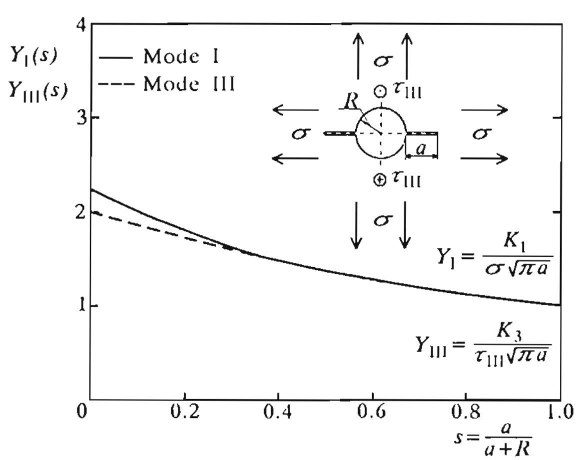

Fig. 5. Correction functions Y_I and Y_{III}, for modes I (solid line) and III (dashed line), for the same normal and tangential stress distribution along the crack path

2.3. Mode III weight functions

A method for obtaining the weight functions forthe mode III is quite different than this described above for modes I and II. It has been observed in Fig.5, presented by Tada et al. (1973), that the stress intensity factor solution for the mode I under equi-biaxial stress state $(\lambda = 1)$ coinsides in the range $0.4 \leq s \leq 1$ with the mode III solution for exactly the same stress distribution along the crack line.

Thus, the weight functions for both modes in that range must be quite similar, from which emerges the conclusion that both the correction functions $F_1(s)$ and $F_3(s)$ as well as all weight coefficients $\Omega_{Ii}(s)$ and $\Omega_{IIIi}(s)$ must be also similar. On the other hand, for $s = 0$ both mode I and mode III solutions coinside with those for a single edge crack in a semi-infinite plate, which are entirely different, but known from the literature. Hence, if asymptotic values on both ends of the domain are known, we can interpolate the remaining range $0 \leq s \leq 0.4$ by a polynomial interpolating formula. These results are represented by dotted lines in Fig.2 and Fig.3.

3. Accuracy assessment

To assess the accuracy of present approach, the calculated stress intensity factors K_1, K_2 and K_3 have been compared to the corresponding solutions taken from the literature (cf Bowie (1956); Newman (1971); Sih (1973); Tada et al. (1973)) and to the BEM results obtained by the author for the mode II. Uniform loads: σ_x and σ_y for mode I, τ for mode II and τ_{III} for mode III (Fig.6), respectively, have been applied to the plate sufficiently far from the cracked area. In the case of mode I – three different loading conditions are analysed:

1. Equi-biaxial tension: $\sigma_x = \sigma_y = \sigma$, $(\lambda = 1)$, where $\lambda = \sigma_x/\sigma_y$

2. Simple tension: $\sigma_x = 0$, $\sigma_y = \sigma$, $(\lambda = 0)$

3. Tension-compresion: $-\sigma_x = \sigma_y = \sigma$, $(\lambda = -1)$.

For all the cases being considered the normal and tangent stress distributions along the potential crack path of uncracked body are well known from the theory of elasticity (cf Timoshenko and Goodier (1951)) and shown in Fig.6.

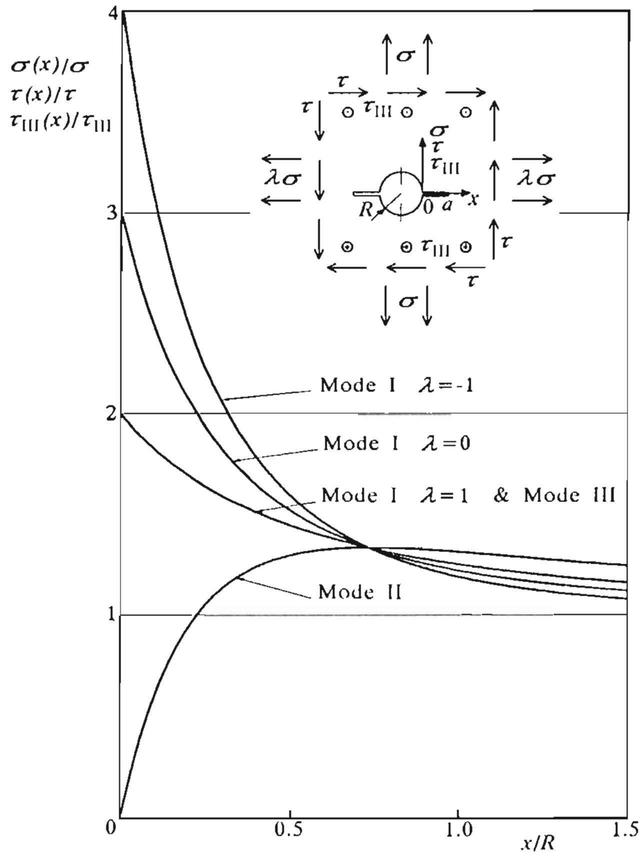

Fig. 6. Elastic stress distributions along the potential crack path of uncracked body, due to various loading conditions

These stresses, together with the weight coefficients $\Omega_i(s)$ and the correction functions $F(s)$ contribute to calculation of stress intensity factors for different a/R ratios. Numerical results obtained using the present approach based on the unitary weight function method (UWF) are compared to the coresponding values of K. The results are shown in Tables 1, 2 and 3 for three different modes, respectively, where three new correction functions $Y_{\rm I}$, $Y_{\rm II}$ and $Y_{\rm III}$ depend on the loading conditions and are defined by the following equations

$$Y_{\rm I} = \frac{K_{\rm I}}{\sigma\sqrt{\pi(a+R)}}$$

$$Y_{II} = \frac{K_2}{\tau\sqrt{\pi(a+R)}} \tag{3.1}$$
$$Y_{III} = \frac{K_3}{\tau_{III}\sqrt{\pi(a+R)}}$$

For all K values shown in the tables, the agreement is very satisfactory with the maximal differences not exceeding one percent.

Table 1. Mode I correction functions $Y_I(a/R)$

a/R	$Y_I(a/R)$ ($\lambda = 1$)		$Y_I(a/R)$ ($\lambda = 0$)		$Y_I(a/R)$ ($\lambda = -1$)	
	UWF	Newman	UWF	Newman	UWF	Newman
0.01	0.2202	0.2188	0.3277	0.3256	0.4352	0.4325
0.02	0.3060	0.3058	0.4517	0.4514	0.5975	0.5971
0.04	0.4182	0.4183	0.6080	0.6082	0.7977	0.7981
0.06	0.4958	0.4958	0.7102	0.7104	0.9246	0.9250
0.10	0.6025	0.6025	0.8393	0.8400	1.0762	1.0775
0.20	0.7484	0.7494	0.9822	0.9851	1.2160	1.2208
0.30	0.8244	0.8259	1.0317	1.0358	1.2391	1.2457
0.50	0.9017	0.9029	1.0546	1.0582	1.2076	1.2134
1.00	0.9684	0.9670	1.0419	1.0409	1.1153	1.1149
1.50	0.9871	0.9855	1.0268	1.0252	1.0666	1.0649
2.00	0.9925	0.9927	1.0162	1.0161	1.0400	1.0395
3.00	0.9932	0.9976	1.0035	1.0077	1.0139	1.0178

Table 2. Mode II correction functions Y_{II}

a/R	$Y_{II}(a/R)$	
	UWF	BEM
0.1	0.1310	0.130
0.2	0.2897	0.289
0.3	0.4267	0.425
0.4	0.5375	0.536
0.5	0.6255	0.623
0.7	0.7504	0.746
1.0	0.8579	0.852
1.2	0.8996	0.892
1.5	0.9382	0.929
2.0	0.9699	0.961
5.0	0.9964	0.988

Table 3. Mode III correction functions Y_{III}

a/R	$Y_{III}(a/R)$	
	UWF	Sih (exact)
0.01	0.1975	0.19753
0.02	0.2758	0.27596
0.04	0.3806	0.38105
0.06	0.4552	0.45597
0.10	0.5619	0.56302
0.20	0.7184	0.71955
0.30	0.8057	0.80615
0.50	0.8968	0.89581
1.00	0.9695	0.96825
1.50	0.9863	0.98712
2.00	0.9910	0.99381

4. Conclusions

The boundary element method together with the complex stress function $Z(z)$ of the Bueckner type singularity at the crack tip have appeared to be a very effective numerical tool in determination of the weight functions of modes I and II for the problem of stress intensity factor calculation for a circular hole with two symmetric radial cracks. Three different weight functions have been found for modes I, II and III, however the solutions for modes I and III coincide in the major part of the range of the shape parameter s. This fact, as well as the unitary weight function approach, made possible the application of assymptotic interpolation to obtaining the solution for anti-plane shear. These three solutions of the weight functions are valid in the whole range of s parameter, i.e. for $0 \leq s \leq 1$, with no singularity found in the correction functions. The accuracy of the stress intensity factors for the three loading modes appeared to be satisfactory with the maximal error much being lower than one percent compared to the results known from the literature and obtained using the boundary element method.

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Funkcje wagowe typu I, II i III dla dwóch symetrycznych szczelin wychodzących z okrągłego otworu

Streszczenie

Rozważano zagadnienie otworu kołowego z dwiema symetrycznymi szczelinami w nieskończonej tarczy, w materiale podlegającym prawu Hooke'a. Wyznaczono wartości funkcji wagowych dla rozciągania oraz ścinania wzdłużnego i poprzecznego. Pierwsze dwa przypadki rozwiązano metodą elementu brzegowego (MEB) w połączeniu z zespoloną funkcją naprężeń, opisującą osobliwość typu Buecknera w wierzchołku szczeliny, natomiast w przypadku ścinania poprzecznego wykorzystano metodę interpolacji asymptotycznej. Otrzymane rozwiązania przedstawiono w formie jednostkowych funkcji wagowych i funkcji korekcyjnych o zunifikowanym zapisie, umożliwiającym utworzenie i wykorzystanie bazy danych, służącej do automatyzacji obliczeń współczynników K przy dowolnym obciążeniu zredukowanym do powierzchni szczeliny. Wartości obliczonych współczynników K_1 i K_3 porównano ze znanymi rozwiązaniami z literatury, otrzymanymi za pomocą innych metod, natomiast wartości K_2 z wynikami MEB. We wszystkich przypadkach różnice te nie przekroczyły jednego procenta.

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