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A GENERALIZATION OF THE INTERNAL VARIABLE MODEL FOR DYNAMICS OF SOLIDS WITH PERIODIC MICROSTUCTURE

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A new micromechanical approach to dynamics of solids with periodic microstructure is proposed. The approach represents a certain generalization of that leading to the refined macrodynamics of micro-periodic composites, [24], by introducing the length scales to the macro-description of both inertial and constitutive properties of a solid.

Key words: microstructure, composites, dynamics

1. Introduction

In order to describe the effect of the microstructure size on the global behaviour of a periodic composite a number of what are called length scale models were proposed (cf [1,2,6,9,16-18]). Models of this kind play an important role mainly in dynamics of solids with the periodic microstructure. That is why in [24] and in the series of related papers [3-5,7,8,10-15,19-35] the modelling of the length scale effect was restricted to the time dependent phenomena for composite solids and structures. In the resulting models the length scale effects were described by the extra unknowns called the macro-internal variables (MIV) and the approach mentioned involved length scales exclusively in the description of non-stationary processes. The aim of this paper is to generalize the micromechanical approach leading to the MIV-model by taking into account the length scale effects also in the description of stationary processes for solids with periodic microstructure. For the sake of simplicity the considerations are restricted to the periodic composites with perfectly bonded constituents and are carried out in the framework of the small displacement gradient theory. It is also assumed that all introduced functions satisfy the regularity conditions required in the subsequent analysis.

Throughout the paper all capital Roman superscripts run over $1, ..., N$ (summation convention holds) if otherwise stated. Points of the physical space E are denoted by $\boldsymbol{x}$, $\boldsymbol{y}$ or $\boldsymbol{z}$ and their distance by $\|\boldsymbol{x}-\boldsymbol{y}\|$. The letter t stands for the time coordinate and $t \in [t_0, t_f]$. By $|\cdot|$ we define both the absolute value of a real number and the length of a vector. Symbol $\mathcal{O}(\varepsilon)$ which in general denotes the set of entities of an order ε, is used exclusively in more restrictive sense described in the subsequent section.

2. Introductory concepts

By Ω we denote a region in the Euclidean 3-space E occupied by the composite solid in the reference configuration. Setting $V := (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ we assume that the solid in this configuration has the V-periodic heterogeneous structure (is V-periodic) and that the micro-structure length parameter defined by $l := \sqrt{l_1^2 + l_2^2 + l_3^2}$ is negligibly small as compared to the smallest characteristic length dimension L_Ω of Ω. We shall use the denotation $V(\boldsymbol{x}) = \boldsymbol{x} + V$; if $V(\boldsymbol{x}) \subset \Omega$ then $V(\boldsymbol{x})$ will be called the cell or the volume element of Ω. The set $\Omega_0 := \{\boldsymbol{x} \in \Omega : V(\boldsymbol{x}) \subset \Omega\}$ is said to be the macro-interior of Ω. For an arbitrary integrable function $f(\cdot)$, defined almost everywhere on Ω, we define the averaged value of $f(\cdot)$ on $V(\boldsymbol{x})$ by means of

$$\langle f(z)\rangle(\boldsymbol{x}) = \frac{1}{l_1 l_2 l_3} \int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\, dv(\boldsymbol{z}) \qquad \boldsymbol{x} \in \Omega_0$$

If $f(\cdot)$ is a V-periodic function then $\langle f(\boldsymbol{z})\rangle(\boldsymbol{x})$ is a constant which will be denoted by $\langle f\rangle$. Now we shall recall two important concepts which will be used in the subsequent analysis.

Let $\Phi(\cdot)$ be a real valued function defined on Ω, which represents a certain scalar field. Let us assume that the values of this field in the problem under consideration have to be calculated (or measured) up to a certain tolerance determined by the tolerance parameter $\varepsilon_\Phi, \varepsilon_\Phi > 0$. It means that an arbitrary real number Φ_0 satisfying condition

$$|\Phi(\boldsymbol{x}) - \Phi_0| < \varepsilon_\Phi$$

can be treated as describing with a sufficient accuracy the value of this scalar field at the point $\boldsymbol{x}$. The triple $(\Phi(\cdot), \varepsilon_\Phi, l)$ will be called the *ε-macrofunction* if the following condition holds

$$\Big(\forall(\boldsymbol{x},\boldsymbol{y}) \in \Omega^2\Big)\Big[\|\boldsymbol{x}-\boldsymbol{y}\| < l \Rightarrow |\Phi(x) - \Phi(y)| < \varepsilon_\Phi\Big]$$

Roughly speaking, from both the calculation and measurement viewpoint, every ε-macrofunction restricted to an arbitrary cell $V(x)$, $\boldsymbol{x} \in \Omega_0$, can be treated as constant. Now assume that $\Phi(\cdot,t)$, $t \in [t_0, t_f]$, is for every t a differentiable function defined on Ω having piecewise continuous time derivatives. Moreover, let Ψ stand for Φ as well as for all derivatives of Φ and assume that every Ψ has to be calculated or measured up to a certain tolerance given by the tolerance parameter ε_Ψ. If every triple $(\Psi(\cdot), \varepsilon_\Psi, l)$ is the ε-macrofunction then the n-tuple $(\Phi(\cdot), \varepsilon_\Phi, \varepsilon_{\nabla\Phi}, \varepsilon_{\dot\Phi}, ..., l)$ is said to be the regular ε-macrofunction. In the sequel we shall tacitly assume that all tolerance parameters $\varepsilon_\Phi, \varepsilon_{\nabla\Phi}, \varepsilon_{\dot\Phi}, ...,$ as well as the microstructure length parameter l are known and refer $\Phi(\cdot)$ to as the regular ε-macrofunction. This concept will be also extended on vector and tensor functions by assuming that all their components in an arbitrary coordinate system are regular ε-macrofunctions.

To the concept of ε-macrofunction certain approximations are strictly related which will be used in this contribution. Let $f(\cdot)$ be an integrable function defined almost everywhere on Ω and $\Phi(\cdot)$ stand for the ε-macrofunction (the tolerance parameter ε_Φ as well as the microstructure length parameter l are assumed to be known). Define by $\mathcal{O}(\varepsilon_\Phi)$ a set of possible local increments $\Delta\Phi$ of Φ such that $|\Delta\Phi| < \varepsilon_\Phi$. Due to the meaning of the ε-macrofunction in the calculation of integrals of the form

$$\int\limits_{V(\boldsymbol{x})} f(z)[\Phi(z) + \mathcal{O}(\varepsilon_\Phi)]\,dv \qquad\qquad \boldsymbol{x} \in \Omega_0$$

terms $\mathcal{O}(\varepsilon_\Phi)$ can be neglected. This statement will be called the *Macro-Averaging Approximation* (MAA). Using the MAA we assign to every $f(\cdot)$ the tolerance relation $\approx$ defined on a set of integrals over $V(x)$ (it is a binary relation which is reflexive and symmetric), given by

$$\int\limits_{V(\boldsymbol{x})} f(z)[\Phi(z) + \mathcal{O}(\varepsilon_\Phi)]\,dv \approx \int\limits_{V(\boldsymbol{x})} f(z)\Phi(z)\,dv \qquad\qquad \boldsymbol{x} \in \Omega_0 \tag{2.1}$$

Since

$$\int\limits_{V(\boldsymbol{x})} f(z)\,dv\,\Phi(x) = \int\limits_{V(\boldsymbol{x})} f(z)[\Phi(z) + \mathcal{O}(\varepsilon_\Phi)]\,dv$$

where now $\mathcal{O}(\varepsilon_\Phi) = \Phi(\boldsymbol{x}) - \Phi(\boldsymbol{z})$ for $\boldsymbol{z} \in V(\boldsymbol{x})$, then Eq (2.1) yields

$$\int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\Phi(\boldsymbol{z})\, dv \approx \int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\, dv\, \Phi(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega_0 \tag{2.2}$$

It has to be emphasized that terms $\mathcal{O}(\varepsilon_\Phi)$ will be neglected only in the course of averaging procedure, i.e., only in the tolerance relations of the form (2.1). Using the denotation $\approx$ in Eqs (2.1) and (2.2) we have tacitly assumed that every tolerance relation $\approx$ is assigned to the integrable function $f(\cdot)$ and is not transitive. It means that in the formulas of the form

$$\int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\Phi_1(\boldsymbol{z})\Phi_2(\boldsymbol{z})\, dv \approx \int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\Phi_1(\boldsymbol{z})\, dv\, \Phi_2(\boldsymbol{x}) \approx \int\limits_{V(\boldsymbol{x})} f(\boldsymbol{z})\, dv\, \Phi_1(\boldsymbol{x})\Phi_2(\boldsymbol{x}) \tag{2.3}$$

where $\Phi_1(\cdot)$, $\Phi_2(\cdot)$ are ε-macrofunctions, the symbols $\approx$ denote two different tolerance relations.

In order to introduce the second fundamental concept used in the subsequent analysis define by $h^A(\cdot)$, $A = 1, 2, ...$, the system of linear independent continuous V-periodic functions (and hence defined on E) having continuous first-order derivatives. Let the above functions satisfy conditions

$$\langle h^A \rangle = \mathbf{0} \qquad \langle h^A h^B \rangle = \delta^{AB} l^2 \qquad (\forall \boldsymbol{x})\Big[|h^A(\boldsymbol{x})| \leq l\Big]$$

and constitute a basis in the space of sufficiently regular functions defined on an arbitrary cell $V(\boldsymbol{x})$ and having on $V(\boldsymbol{x})$ the averaged values equal to zero. Under the aforementioned conditions the system $h^A(\cdot)$, $A = 1, 2, ...$, will be called *the local oscillation basis.*

The concepts of the regular ε-macrofunction and the local oscillation basis as well as the macro-averaging approximation (MAA) formulated above constitute the fundamentals of the micromechanical approach to the macrodynamics of composites which will be proposed in this contribution.

3. Kinematics

Let $\boldsymbol{u}(\cdot, t)$ stand for a displacement field defined on Ω for every instant t. Define on Ω_0 the averaged displacement fields by means of

$$\begin{aligned} \boldsymbol{U}(\boldsymbol{x}, t) &:= \langle \boldsymbol{u}(\boldsymbol{z}, t)\rangle(\boldsymbol{x}) \\ \boldsymbol{W}^A(\boldsymbol{x}, t) &:= \langle [\boldsymbol{u}(\boldsymbol{z}, t) - \boldsymbol{U}(\boldsymbol{z}, t)] h^A(\boldsymbol{z})\rangle(\boldsymbol{x}) l^{-2} \qquad \boldsymbol{x} \in \Omega_0 \end{aligned} \tag{3.1}$$

where $A = 1, 2, \ldots$. Moreover, let $\boldsymbol{r_x}(\cdot, t)$ be an arbitrary regular vector field defined on $V(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_0$, satisfying condition

$$\langle \boldsymbol{r_x}(z, t)\rangle(\boldsymbol{x}) = \langle \boldsymbol{U}(z, t)\rangle(\boldsymbol{x}) - \boldsymbol{U}(\boldsymbol{x}, t) \tag{3.2}$$

The form of $\boldsymbol{r_x}(\cdot, t)$ will be specified in the subsequent part of this section. By the local displacement oscillations we shall mean the vector functions $\boldsymbol{w_x}(\cdot, t)$ defined on $V(\boldsymbol{x})$ and given by

$$\boldsymbol{w_x}(\boldsymbol{y}, t) = \boldsymbol{u}(\boldsymbol{y}, t) - \boldsymbol{U}(\boldsymbol{y}, t) + \boldsymbol{r_x}(\boldsymbol{y}, t) \qquad \boldsymbol{y} \in V(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_0 \tag{3.3}$$

The above fields satisfy conditions $\langle w_{\boldsymbol{x}}(z, t)\rangle(\boldsymbol{x}) = \boldsymbol{0}$ and under the known regularity conditions can be represented by

$$\boldsymbol{w_x}(\boldsymbol{y}, t) = \sum_{A=1}^{\infty} [W^A(\boldsymbol{x}, t) + \langle \boldsymbol{r_x}(z, t) h^A(z)\rangle(\boldsymbol{x}) l^{-2}] h^A(\boldsymbol{y}) \tag{3.4}$$

The kinematics of the solids with periodic microstructure will be based on the MAA (cf Section 2) and on two following assumptions.

Truncation Assumption (TA) states that the Fourier series (3.4) can be approximated by the sum of the first N terms, $N \geq 1$, where N has to be specified in every problem under consideration.

Kinematic Macro-Regularity Assumption (KMRA) restricts the class of motions we are to investigate by assuming that the fields $\boldsymbol{U}(\cdot, t)$, $\boldsymbol{W}^A(\cdot, t)$, $A = 1, 2, \ldots, N$, are regular ε-macrofunctions.

Bearing in mind the aforementioned assumptions we shall refer fields $\boldsymbol{U}(\cdot, t)$, $\boldsymbol{W}^A(\cdot, t)$, $A = 1, 2, \ldots, N$, to as the macrodisplacements and the oscillation variables, respectively. Moreover, functions $h^A(\cdot)$, $A = 1, \ldots, N$ will be called the micro-shape functions. In the sequel all capital superscripts run over $1, \ldots, N$; the summation convention holds. The kinematics of the micro-periodic solids under consideration will be summarized in the following statement.

Lemma. Under TA, KMRA and using MAA it can be assumed that the displacement fields $\boldsymbol{u}(\cdot, t)$ are related to the macrodisplacements $\boldsymbol{U}(\cdot, t)$ and the oscillation variables $W^A(\cdot, t)$ by means of the formula

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{x}, t) + h^A(\boldsymbol{x}) W^A(\boldsymbol{x}, t) \qquad \boldsymbol{x} \in \Omega_0 \tag{3.5}$$

which has to hold for every time t.

In order to prove this lemma let us specify $\boldsymbol{r}_{\boldsymbol{x}}(\cdot,t)$ to the form

$$\boldsymbol{r}_{\boldsymbol{x}}(\boldsymbol{y},t) := h^A(\boldsymbol{y})[\boldsymbol{W}^A(\boldsymbol{x},t) - \boldsymbol{W}^A(\boldsymbol{y},t)] \qquad \boldsymbol{y} \in V(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega_0 \tag{3.6}$$

By means of Eqs (3.2) and $(3.1)_1$ we obtain

$$\langle \boldsymbol{u}(z,t) - \boldsymbol{U}(z,t) - h^A(z)\boldsymbol{W}^A(z,t)\rangle(\boldsymbol{x}) = \boldsymbol{0} \qquad \boldsymbol{x} \in \Omega_0 \tag{3.7}$$

From Eqs (3.3), (3.4) and TA it follows that

$$\boldsymbol{u}(\boldsymbol{y},t) = \boldsymbol{U}(\boldsymbol{y},t) + h^A(\boldsymbol{y})\boldsymbol{W}^A(\boldsymbol{y},t) + \langle \boldsymbol{r}_{\boldsymbol{x}}(z,t)h^A(z)\rangle(\boldsymbol{x})l^2h^A(\boldsymbol{y}) \tag{3.8}$$

for $\boldsymbol{y} \in V(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_0$. Similarly

$$\begin{aligned} \nabla\boldsymbol{u}(\boldsymbol{y},t) &= \nabla\boldsymbol{U}(\boldsymbol{y},t) + \nabla[h^A(\boldsymbol{y})\boldsymbol{W}^A(\boldsymbol{y},t)] + \langle \boldsymbol{r}_{\boldsymbol{x}}(z,t)h^A(z)\rangle(\boldsymbol{x})l^2\nabla h^A(\boldsymbol{y}) \\ {}^{(n)}\boldsymbol{u}(\boldsymbol{y},t) &= {}^{(n)}\boldsymbol{U}(\boldsymbol{y},t) + h^A(\boldsymbol{y})\,{}^{(n)}\boldsymbol{W}^A(\boldsymbol{y},t) + \langle {}^{(n)}\boldsymbol{r}_{\boldsymbol{x}}(z,t)h^A(z)\rangle(\boldsymbol{x})l^2h^A(\boldsymbol{y}) \\ & \qquad n = 1,2 \end{aligned} \tag{3.9}$$

where the superscript (n) stands for the nth order time derivative. Using MAA in the form given by Eq (2.2) we have

$$\begin{aligned} \langle \boldsymbol{r}_{\boldsymbol{x}}(z,t)h^A(z)\rangle(\boldsymbol{x}) &= \langle h^A(z)h^B(z)[\boldsymbol{W}^B(\boldsymbol{x},t) - \boldsymbol{W}^B(z,t)]\rangle(\boldsymbol{x}) \approx \\ &\approx \langle h^A(z)h^B(z)\rangle(\boldsymbol{x})[\boldsymbol{W}^B(\boldsymbol{x},t) - \boldsymbol{W}^B(\boldsymbol{x},t)] = \boldsymbol{0} \end{aligned}$$

Because of the KMRA and definition (3.6), in the framework of MMA the terms $\langle \boldsymbol{r}_{\boldsymbol{x}}(z,t)h^A(z)\rangle(\boldsymbol{x})$ in Eqs (3.8), $(3.9)_1$ and their time derivatives in $(3.9)_2$ can be neglected. It follows that Eq (3.5) holds true and hence condition (3.2) is identically satisfied, which ends the proof.

4. Dynamics

Let $\boldsymbol{s}(\cdot,t)$ stand for the Cauchy stress tensor field defined for every time instant t. Define on Ω_0 the following averaged fields

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{x},t) &:= \langle \boldsymbol{s}(z,t)\rangle(\boldsymbol{x}) \\ \boldsymbol{H}^A(\boldsymbol{x},t) &:= \langle \boldsymbol{s}(z,t)\cdot\nabla h^A(z)\rangle(\boldsymbol{x}) \\ \boldsymbol{R}^A(\boldsymbol{x},t) &:= \langle \boldsymbol{s}(z,t)h^A(z)\rangle(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Omega_0 \end{aligned} \tag{4.1}$$

The passage from micro- to macrodynamics for the solids with periodic microstructure will be based on the macro-averaging approximation, on the lemma formulated in Section 3 as well as on the following assumption.

Stress Macro-Regularity Assumption (SMRA). The stress distribution in the problems under consideration is restricted by the condition that the fields $\boldsymbol{S}(\cdot,t)$, $\boldsymbol{H}^A(\cdot,t)$, $\boldsymbol{R}^A(\cdot,t)$, $A=1,...,N$, are regular ε-macrofunctions.

Let $\rho(\cdot)$ stand for the mass density field (which is a V-periodic function defined almost everywhere on Ω) and assume that the body force $\boldsymbol{b}$ is constant. Let us denote by $\boldsymbol{n}(\boldsymbol{y})$ the unit normal outward to $\partial V(\boldsymbol{x})$ at $\boldsymbol{y}$. The starting point of the proposed micromechanical procedure will be the weak form of equations of motion of micromechanics. Taking into account the symmetry of the stress tensor these equations can be assumed in the form of conditions

$$\int\limits_{V(\boldsymbol{x})} \boldsymbol{s}(\boldsymbol{y},t) : \nabla\overline{\boldsymbol{u}}(\boldsymbol{y})\, dv = \oint\limits_{\partial V(\boldsymbol{x})} [\boldsymbol{s}(\boldsymbol{y},t)\cdot\boldsymbol{n}(\boldsymbol{y})]\cdot\overline{\boldsymbol{u}}(\boldsymbol{y})\, da + \int\limits_{V(\boldsymbol{x})} \rho(\boldsymbol{y})[\boldsymbol{b}-\ddot{\boldsymbol{u}}(\boldsymbol{y},t)]\cdot\overline{\boldsymbol{u}}(\boldsymbol{y})\, dv \tag{4.2}$$

which have to hold for every $\boldsymbol{x}\in\Omega_0$ and for an arbitrary test function $\overline{\boldsymbol{u}}(\cdot)$. By means of Eq (3.5) we assume that

$$\overline{\boldsymbol{u}}(\boldsymbol{y}) = \overline{\boldsymbol{U}}(\boldsymbol{y}) + h^A(\boldsymbol{y})\overline{\boldsymbol{W}}^A(\boldsymbol{y}) \qquad \boldsymbol{y}\in\Omega \tag{4.3}$$

where $\overline{\boldsymbol{U}}(\cdot), \overline{\boldsymbol{W}}^A(\cdot)$ are arbitrary linear-independent regular ε-macrofunctions.

Now we shall formulate the fundamental assertion of the micromechanical approach to macrodynamics proposed in this contribution. For the sake of simplicity we shall also assume that the micro-shape functions $h^A(\cdot)$ satisfy the extra conditions $\langle\rho h^A\rangle = 0$, $A=1,...,N$. We also assume that the piecewise constant (discontinuous) distribution of heterogeneity is approximated in the vicinity of interfaces by a continuous one. Hence, in the subsequent analysis the stress continuity conditions across interfaces will be not taken into account.

Basic Assertion. Under TA, KMRA, SMRA and in the framework of MAA the equations of motion (4.2), (4.3) imply the following interrelation between the fields defined by Eqs (3.1) and (4.1)

$$\begin{aligned} &\operatorname{div}\boldsymbol{S}(\boldsymbol{x},t) - \langle\rho\rangle\ddot{\boldsymbol{U}}(\boldsymbol{x},t) + \langle\rho\rangle\boldsymbol{b} = \boldsymbol{0} \\ &-\operatorname{div}\boldsymbol{R}^A(\boldsymbol{x},t) + \langle\rho h^A h^B\rangle\ddot{\boldsymbol{W}}^B(\boldsymbol{x},t) + \boldsymbol{H}^A(\boldsymbol{x},t) = \boldsymbol{0} \end{aligned} \tag{4.4}$$

which holds for every $\boldsymbol{x}\in\Omega_0$ and every t.

In order to prove the above assertion let us combine Eqs (4.2) and (4.3). Since

$$\int\limits_{V(\boldsymbol{x})} \boldsymbol{s} : \nabla\overline{U}\, dv \approx \int\limits_{V(\boldsymbol{x})} \boldsymbol{s} dv : \nabla\overline{U}(\boldsymbol{x}) \approx \int\limits_{V(\boldsymbol{x})} \boldsymbol{S} : \nabla\overline{U}\, dv =$$

$$- \int\limits_{V(\boldsymbol{x})} \mathrm{div}\boldsymbol{S}\cdot\overline{U}\, dv + \oint\limits_{\partial V(\boldsymbol{x})} (\boldsymbol{S}\cdot\boldsymbol{n})\cdot\overline{U}\, da$$

$$\int\limits_{V(\boldsymbol{x})} h^A\boldsymbol{s} : \nabla\overline{\boldsymbol{W}}^A\, dv \approx \int\limits_{V(\boldsymbol{x})} \boldsymbol{s}h^A dv : \nabla\overline{\boldsymbol{W}}^A(\boldsymbol{x}) \approx \int\limits_{V(\boldsymbol{x})} \boldsymbol{R}^A : \nabla\overline{\boldsymbol{W}}^A\, dv =$$

$$= - \int\limits_{V(\boldsymbol{x})} \mathrm{div}\boldsymbol{R}^A\cdot\overline{\boldsymbol{W}}^A\, dv + \oint\limits_{\partial V(\boldsymbol{x})} (\boldsymbol{R}^A\cdot\boldsymbol{n})\cdot\overline{\boldsymbol{W}}^A\, da$$

$$\int\limits_{V(\boldsymbol{x})} (\boldsymbol{s}\cdot\nabla h^A)\cdot\overline{\boldsymbol{W}}^A\, dv \approx \int\limits_{V(\boldsymbol{x})} \boldsymbol{H}^A\cdot\overline{\boldsymbol{W}}^A\, dv$$

$$\int\limits_{V(\boldsymbol{x})} \rho(\boldsymbol{b}-\ddot{\boldsymbol{u}})\cdot\overline{U}\, dv = \int\limits_{V(\boldsymbol{x})} \Big(\langle\rho\rangle\boldsymbol{b} - \langle\rho\rangle\ddot{U}\Big)\cdot\overline{U}\, dv$$

$$\int\limits_{V(\boldsymbol{x})} h^A\rho(\boldsymbol{b}-\ddot{\boldsymbol{u}})\cdot\overline{\boldsymbol{W}}^A\, dv = - \int\limits_{V(\boldsymbol{x})} \langle\rho h^A h^B\rangle\ddot{\boldsymbol{W}}^B\cdot\overline{\boldsymbol{W}}^A\, dv$$

then using the MAA we have

$$\int\limits_{V(\boldsymbol{x})} \Big(\mathrm{div}\boldsymbol{S} - \langle\rho\rangle\ddot{U} + \langle\rho\rangle\boldsymbol{b}\Big)\cdot\overline{U}\, dv - \oint\limits_{\partial V(\boldsymbol{x})} [(\boldsymbol{S}-\boldsymbol{s})\cdot\boldsymbol{n}]\cdot\overline{U}\, da = 0$$

$$\int\limits_{V(\boldsymbol{x})} \Big(-\mathrm{div}\boldsymbol{R}^A + \langle\rho h^A h^B\rangle\ddot{\boldsymbol{W}}^B + \boldsymbol{H}^A\Big)\cdot\overline{\boldsymbol{W}}^A\, dv + \tag{4.5}$$

$$+ \oint\limits_{\partial V(\boldsymbol{x})} \Big[(\boldsymbol{R}^A - \boldsymbol{s}h^A)\cdot\boldsymbol{n}\Big]\cdot\overline{\boldsymbol{W}}^A\, da = 0$$

Bearing in mind that $V(\boldsymbol{x}) = V + \boldsymbol{x}$ and introducing the local coordinate $\boldsymbol{y}\in V$ we also obtain

$$\oint\limits_{\partial V(\boldsymbol{x})} (\boldsymbol{s}\cdot\boldsymbol{n})\cdot\overline{U}\, da = \int\limits_{V} \mathrm{div}(\boldsymbol{s}\cdot\overline{U})\, dv \approx \int\limits_{V} \mathrm{div}_{\boldsymbol{x}}[\boldsymbol{s}(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{U}(\boldsymbol{x})]\, dv(\boldsymbol{y}) \approx$$

$$\mathrm{div}_{\boldsymbol{x}} \int\limits_{V} \boldsymbol{S}(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{U}(\boldsymbol{x})\, dv(\boldsymbol{y}) \approx \int\limits_{V} \mathrm{div}_{\boldsymbol{x}}[\boldsymbol{S}(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{U}(\boldsymbol{x}+\boldsymbol{y})]\, dv(\boldsymbol{y}) =$$

$$= \int\limits_{V(\boldsymbol{x})} \mathrm{div}(\boldsymbol{S}\cdot\overline{U})\, dv$$

$$\oint\limits_{\partial V(\boldsymbol{x})} h^A(\boldsymbol{s}\cdot\boldsymbol{n})\cdot\overline{\boldsymbol{W}}^A\, da = \int\limits_{V(\boldsymbol{x})} \operatorname{div}(h^A\boldsymbol{s}\cdot\overline{\boldsymbol{W}}^A)\, dv \approx$$

$$\approx \int\limits_{V} \operatorname{div}_{\boldsymbol{x}}[h^A(\boldsymbol{x}+\boldsymbol{y})\boldsymbol{s}(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{\boldsymbol{W}}^A(\boldsymbol{x})]\, dv(\boldsymbol{y}) =$$

$$= \operatorname{div}_{\boldsymbol{x}} \int\limits_{V} \boldsymbol{R}^A(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{\boldsymbol{W}}^A(\boldsymbol{x})\, dv(\boldsymbol{y}) \approx$$

$$\approx \int\limits_{V} \operatorname{div}_{\boldsymbol{x}}[\boldsymbol{R}^A(\boldsymbol{x}+\boldsymbol{y},t)\cdot\overline{\boldsymbol{W}}^A(\boldsymbol{x}+\boldsymbol{y})]\, dv(\boldsymbol{y}) = \int\limits_{V(\boldsymbol{x})} \operatorname{div}(\boldsymbol{R}^A\cdot\overline{\boldsymbol{W}}^A)\, dv$$

and by virtue of the MAA the surface integrals in Eqs (4.5) can be neglected. Hence formulae (4.5) imply Eqs (4.4), which ends the proof.

5. Constitutive relations

It has to be emphasized that Eqs (4.4) have been derived without any reference to the material properties of the solid under consideration. In order to obtain the complete set of the field equations for the description of solids with microstructure on the macro-level also we have to derive the constitutive equations. It can be done for an arbitrary simple material but for the sake of simplicity we shall confine ourselves to the composites made of the linear-elastic constituents. Setting $\boldsymbol{s} = \boldsymbol{C}(\boldsymbol{z}) : \boldsymbol{e}$, where $\boldsymbol{e} = 0.5[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\top}]$ and $\boldsymbol{C}(\cdot)$ is the V-periodic piecewise constant elasticity tensor field, under the denotation

$$\boldsymbol{E} := \frac{1}{2}\Big[\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^{\top}\Big] \tag{5.1}$$

from Eqs (4.1) by using the MAA we obtain

$$\begin{aligned}
\boldsymbol{S}(\boldsymbol{x},t) &= \langle\boldsymbol{C}\rangle : \boldsymbol{E}(\boldsymbol{x},t) + \langle\boldsymbol{C}\cdot\nabla h^A\rangle\cdot\boldsymbol{W}^A(\boldsymbol{x},t) + \underline{\langle\boldsymbol{C}h^A\rangle} : \nabla\boldsymbol{W}^A(\boldsymbol{x},t)\\
\boldsymbol{H}^A(\boldsymbol{x},t) &= \langle\nabla h^A\cdot\boldsymbol{C}\rangle : \boldsymbol{E}(\boldsymbol{x},t) + \langle\nabla h^A\cdot\boldsymbol{C}\cdot\nabla h^B\rangle\cdot\boldsymbol{W}^B(\boldsymbol{x},t) +\\
&\quad + \underline{\langle\nabla h^A\cdot\boldsymbol{C}h^B\rangle} : \nabla\boldsymbol{W}^B(\boldsymbol{x},t)\\
\boldsymbol{R}^A(\boldsymbol{x},t) &= \underline{\langle\boldsymbol{C}h^A\rangle} : \boldsymbol{E}(\boldsymbol{x},t) + \underline{\langle h^A\boldsymbol{C}\cdot\nabla h^B\rangle}\cdot\boldsymbol{W}^B(\boldsymbol{x},t) +\\
&\quad + \underline{\langle h^A h^B\boldsymbol{C}\rangle} : \nabla\boldsymbol{W}^B(\boldsymbol{x},t)
\end{aligned} \tag{5.2}$$

The equations of motion (4.4) together with the constitutive equations (5.2) and the denotations (5.1) constitute the model of the linear-elastic composite solid with periodic microstructure. Since the underlined material moduli in Eqs (5.2) as well as the inertial moduli $\langle \rho h^A h^B \rangle$ in Eqs (4.4) depend on the microstructure size (since $h^A(z) \in \mathcal{O}(l)$) then we have obtained the class of length-scale models the form of which is determined by the choice of micro-shape functions $h^A(\cdot)$, $A = 1,...,N$. These models constitute a certain generalization of models with the internal variables (MIV-models), in which the underlined terms in Eqs (5.2) (and hence also the term $\mathrm{div}\boldsymbol{R}^A$ in Eqs (4.4)) are absent, cf [23] and the related papers quoted in Introduction.

6. Passage to MIV-models

Let us assume that every function $h^A(\cdot)$ satisfies the condition

$$(\forall \boldsymbol{x} \in \Lambda_A)\Big[h^A\Big|_{\partial V(\boldsymbol{x})} = 0\Big] \qquad A = 1,2,... \tag{6.1}$$

where Λ_A is a certain V-periodic discrete lattice of points in E. Such situation takes place, e.g., for the Fourier expansions (3.4) given by the trygonometric series. Now we are to show that if all microshape functions satisfy Eq (6.1) then the model given by Eqs (4.4), (5.2) reduces to the model with macro-internal variables. The proof of this statement will consist of two independent parts.

First, let us observe that by means of $V(\boldsymbol{x}) = \boldsymbol{x} + V$ and denoting by $\boldsymbol{y} \in V$ the local coordinate, we obtain

$$\begin{aligned} \mathrm{div}\boldsymbol{R}^A(\boldsymbol{x},t) &= \mathrm{div}\langle s(z,t)h^A(z)\rangle(\boldsymbol{x}) = \mathrm{div}_{\boldsymbol{x}}\langle s(\boldsymbol{x}+\boldsymbol{y},t)h^A(\boldsymbol{x}+\boldsymbol{y})\rangle(\boldsymbol{0}) = \\ &= \langle \mathrm{div}_{\boldsymbol{x}}[s(\boldsymbol{x}+\boldsymbol{y},t)h^A(\boldsymbol{x}+\boldsymbol{y}))(\boldsymbol{0}) = \langle \mathrm{div}[s(z,t)h^A(z)]\rangle(\boldsymbol{x}) = \\ &= \frac{1}{|V|}\oint_{\partial V(\boldsymbol{x})} h^A \boldsymbol{s}\cdot\boldsymbol{n}\,da \qquad |V| \equiv l_1 l_2 l_3 \end{aligned} \tag{6.2}$$

Since $h^A(\cdot)$ satisfies Eq (6.1) then the conditions $\mathrm{div}\boldsymbol{R}^A(\boldsymbol{x},t) = \boldsymbol{0}$ hold for every $\boldsymbol{x} \in \Lambda_A \cap \Omega_0$. At the same time from the SMRA it follows that $\mathrm{div}\boldsymbol{R}^A(\cdot,t)$ is the ε-macrofunction. Hence, for every $\boldsymbol{z} \in V(\boldsymbol{x})$

$$\int_{V(\boldsymbol{x})} [\mathrm{div}\boldsymbol{R}^A(\boldsymbol{x},t) - \mathrm{div}\boldsymbol{R}^A(z,t)]\,dv \approx \boldsymbol{0} \qquad \boldsymbol{x} \in \Omega_0$$

Setting $\boldsymbol{x} \in \Lambda_A$ into the above relation we obtain $\mathrm{div}\boldsymbol{R}^A(\boldsymbol{z},t) \approx \boldsymbol{0}$ for every $\boldsymbol{z} \in \Omega_0$. Thus, we have arrived at the conclusion that Eqs (4.4) in the framework of MAA reduce to the form

$$\begin{aligned} &\mathrm{div}\boldsymbol{S}(\boldsymbol{x},t) - \langle\rho\rangle\ddot{\boldsymbol{U}}(\boldsymbol{x},t) + \langle\rho\rangle\boldsymbol{b} = \boldsymbol{0} \\ &\langle\rho h^A h^B\rangle\ddot{\boldsymbol{W}}^B(\boldsymbol{x},t) + \boldsymbol{H}^A(\boldsymbol{x},t) = \boldsymbol{0} \end{aligned} \tag{6.3}$$

which coincides with that of the equations of motion and the dynamic evolution equation of the MIV-model.

Second, we shall prove that under (6.1) the following formula holds for an arbitrary sufficiently regular V-periodic tensor field $\boldsymbol{F}(\cdot)$

$$\langle\boldsymbol{F}\cdot\nabla(h^A\boldsymbol{W}^A)\rangle(\boldsymbol{x}) \approx \langle\boldsymbol{F}\cdot\nabla h^A\rangle \otimes \boldsymbol{W}^A(\boldsymbol{x}) \tag{6.4}$$

To this end let us observe that

$$\langle\boldsymbol{F}\cdot\nabla h^A\rangle(\boldsymbol{x}) = \frac{1}{|V|}\int\limits_{\Gamma(\boldsymbol{x})} h^A[\![\boldsymbol{F}]\!]\cdot\boldsymbol{n}\;da - \langle h^A\mathrm{div}\boldsymbol{F}\rangle(\boldsymbol{x}) \tag{6.5}$$

where $[\![\boldsymbol{F}]\!]$ is a jump of $\boldsymbol{F}$ across all interfaces $\Gamma(\boldsymbol{x})$, oriented by a unit normal $\boldsymbol{n}$ in $V(\boldsymbol{x})$. At the same time for every A we obtain (no summation over A!)

$$\begin{aligned} &\langle\boldsymbol{F}\cdot\nabla(h^A\boldsymbol{W}^A)\rangle(\boldsymbol{x}) = \langle\mathrm{div}(\boldsymbol{F}\otimes\boldsymbol{W}^A h^A)\rangle(\boldsymbol{x}) - \langle\mathrm{div}\boldsymbol{F}\otimes\boldsymbol{W}^A h^A\rangle \approx \\ &\approx \frac{1}{|V|}\oint\limits_{\partial V(\boldsymbol{x})} h^A(\boldsymbol{F}\cdot\boldsymbol{n})\otimes\boldsymbol{W}^A\;da + \\ &+\Big(\frac{1}{|V|}\int\limits_{\Gamma(\boldsymbol{x})} h^A[\![\boldsymbol{F}]\!]\cdot\boldsymbol{n}\;da - \langle h^A\mathrm{div}\boldsymbol{F}\rangle(\boldsymbol{x})\Big)\otimes\boldsymbol{W}^A(\boldsymbol{x},t) \end{aligned} \tag{6.6}$$

For every $\boldsymbol{x} \in \Lambda_A$ the value of the first from integrals on the right-hand side of Eq (6.6) is equal to zero. At the same time this integral represents a certain ε-macrofunction defined on Ω_0 (since $\boldsymbol{W}^A(\cdot,t)$ is the ε-macrofunction) and hence bearing in mind Eq (6.5), we conclude that Eq (6.4) holds true. Using this result we also have

$$\langle\boldsymbol{F}\cdot\nabla h^A\rangle\cdot\boldsymbol{W}^A(\boldsymbol{x},t) + \langle\boldsymbol{F}h^A\rangle : \nabla\boldsymbol{W}^A(\boldsymbol{x},t) \approx \langle\boldsymbol{F}\cdot\nabla h^A\rangle\cdot\boldsymbol{W}^A(\boldsymbol{x},t)$$

and hence Eqs (5.2) in the framework of MAA reduce to the following ones

$$\begin{aligned} &\boldsymbol{S}(\boldsymbol{x},t) = \langle\boldsymbol{C}\rangle : \boldsymbol{E}(\boldsymbol{x},t) + \langle\boldsymbol{C}\cdot\nabla h^A\rangle\cdot\boldsymbol{W}^A(\boldsymbol{x},t) \\ &\boldsymbol{H}^A(\boldsymbol{x},t) = \langle\nabla h^A\cdot\boldsymbol{C}\rangle : \boldsymbol{E}(\boldsymbol{x},t) + \langle\nabla h^A\cdot\boldsymbol{C}\cdot\nabla h^B\rangle\cdot\boldsymbol{W}^B(\boldsymbol{x},t) \end{aligned} \tag{6.7}$$

The above equations are the macro-constitutive equations of the linear-elastic composites for the MIV-model.

7. Conclusions

The main conclusion is that if all micro-shape functions $h^A(\cdot)$, $A = 1, ..., N$, satisfy conditions (6.1) then the generalized model of composites derived in this paper and given by Eqs (4.4), (5.2) reduces to the known model with internal macro-variables determined by Eqs (6.3), (6.7). If conditions (6.1) do not hold then we have to use more general equations (4.4), (5.2) in which the length scale effect on the global body behaviour takes place also in the stationary problems. Since the choice of micro-shape functions (i.e. the choice of the local oscillation basis and the truncation assumption) determines a class of micro-motions which are assumed to be relevant in the problem under consideration, then the effect of microstructure size also depends on the micro-motions we are to investigate. It has to be emphasized that if conditions (6.1) do not hold then in formulations of the boundary value problems for Eqs (4.4), (5.2) the boundary conditions have to be postulated not only for $\boldsymbol{U}$ but also on the extra unknowns $\boldsymbol{W}^A$. Such situation does not take place for the boundary-value problems analyzed in the framework of MIV-models where we deal only with three boundary conditions for three components of $\boldsymbol{U}$. The detailed discussion of the models presented in this contribution as well as their comparison will be given in a separate paper.

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Uogólnienie modelu z wewnętrznymi zmiennymi w dynamice ciał o periodycznej mikrostrukturze

Streszczenie

W pracy przedstawiono nowe mikromechaniczne podejście do dynamiki ciał o periodycznej mikrostrukturze. Otrzymany model jest pewnym uogólnieniem ulepszonej makrodynamiki mikroperiodycznych kompozytów, [24], uwzględniając efekt wielkości mikrostruktury w opisie zarówno inercjalnych jak i materiałowych własności ciała.

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