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THE INFLUENCE OF DIMENSIONLESS PARAMETER ORDERS OF MAGNITUDE ON THE HOMOGENISATION PROCESS RESULT

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Basing on two examples a significant impact of the order of magnitude of dimensionless parameters on the homogenisation process results when using either the direct averaging or classical homogenisation approach has been presented. First, the effect of two dimensionless parameters on the results of amplitude direct homogenisation procedure has been studied. Three examples of different values of dimensionless parameters Q and R_{el} have been shown for filtration of an incompressible Newtonian liquid. The presented methodology proves that certain classes of physical processes can be treated in a similar way. Dissimilar descriptions of the processes appearing in, for instance optics or dynamics result from neglecting certain effects when changing the observation method or scale.

Key words: homogenisation, dynamics, fluid flow

1. Introduction

The homogenisation theory is usually applied when one has to find an equivalent homogeneous model of a heterogeneous medium under consideration. An overwhelming majority of material solids which continuum mechanics is interested in reveal the feature of material properties heterogeneity displaying a random spatial distribution. Only few materials, usually the artificial ones, exhibit periodic distributions of heterogeneity. In this paper, we shall use the direct homogenisation and the method of averaging based on the theory of small parameter, applied to materials of periodic heterogeneity distribution. This is for the following two reasons. Firstly, there is evidence for equivalence of the two methods of homogenisation of randomly and periodically heterogeneous media (Kröner, 1972; Strzelecki et al., 1996). Secondly, the method of homogenisation of deterministically heterogeneous medium can be followed in much more easy way when taking into account the necessary mathematical apparatus. Periodic heterogeneity is assumed to oscillate at a high frequency within an area of the body under consideration when travelling from point to point. It appears that considerable or even jump changes of the medium properties that occur on a small scale (microscale) are observed as large-scale (macroscale) variability imposed on the trend function of the physical value we are interested in. These distortions tend to vanish in the course of homogenisation process when the medium is homogenisable or contribute considerably to solutions to the non-homogenisable problems.

Whether or not a heterogeneous medium is homogenisable, or if there exists an equivalent model of the medium and a theory that describes it, is determined by the orders of magnitude of dimensionless parameters that appear in the equations written in dimensionless coordinates. These parameters determine the processes in a heterogeneity cell (Auriault, 1986). There are two parameters in the steady-state vibrations, considered below, one of which being the basic *epsilon* parameter of homogenisation. When analysing liquid steady flows through a porous medium, apart from the *epsilon* parameter, also two dimensionless parameters are used. If, for a given boundary-value problem, the heterogeneous medium is homogenisable, it is possible to find the effective material constants for a homogeneous simplified model. Then these constants are used in the simplified equation describing the physical value being looked for. The solution to this problem is a function, being the limit the solution obtained for a periodic homogeneous medium tends to approach. We obtain this limit when we approach zero with the cell size. This also means that, when the periodic heterogeneity "cell" is small enough when compared to the size of the body in question, the solution precisely taking into account heterogeneities differs slightly from the solution for homogenisation-simplified model of the medium. In addition, the theory of homogenisation allows us to determine in which boundary-value problems the fluctuations induced by the body structure heterogeneity are so big that they constitute a part of the solution that cannot be neglected. Such issues are non-homogenisable, hence approximate, simpler, averaged material constants solutions obtained for models of homogeneous media do not exist. In the theory of homogenisation, it is often assumed that not only the size of the periodic heterogeneity cell tends to zero. It is also assumed that the other parameters that describe the problem, including some physical constants, might change in proportion to the ε^a-function of the small *epsilon* parameter associated with the material constants periodicity cell (Auriault and Royer, 1993).

The α-power exponent might be any real number, but usually takes one of the following values: -2, -1, $-1/2$, 0, 1/2, 1, 2. The physical sense of the assumption that the values of physical constants should change alongside with a change of ε is sometimes not clear. It is obvious only in the direct method of homogenisation. In this method, we know a precise closed form solution to the problem for a periodically heterogeneous medium where, by a direct limit passage to zero with the parameter ε, we get a simplified solution, valid for a medium with the effective material constants. If the problem is not homogenisable, this limit does not exist. However, the method of direct homogenisation displays one basic drawback. There are hardly any closed form solutions for heterogeneous media, important from the technical viewpoint. Using this method, it is impossible to obtain the equation representing the problem for the equivalent homogeneous medium. On the other hand, the method of direct homogenisation is of great importance for didactic purposes owing to its clear mathematical and physical sense. It is a tool for understanding the idea of homogenisation and makes it easier to get skilled at other, more general, methods of averaging. Therefore, below we shall present the analysis of the effect the values of dimensionless parameters exert on the result of the averaging process, following the direct homogenisation procedure, in a simple 1D mechanical problem of the dynamic theory of elasticity.

A similar problem of the effect the values of dimensionless parameters exert on homogenisation result is presented while averaging an incompressible liquid flow through the pores of a heterogeneous medium. However, unlike in the former case, the way of homogenisation is based on a general method, in which the small parameter approach is employed.

2. Samples of direct homogenisation of the solutions within the framework of dynamic theory of elasticity

The boundary problems of the dynamic elasticity theory of heterogeneous media is described by the equation

$$\Big(G(U_{i,j}+U_{j,i})\Big)_{,j}+\Big(\lambda U_{j,j}\Big)_{,i}+\gamma_i=\rho\frac{\partial^2}{\partial t^2}U_i \tag{2.1}$$

For clarity, in the above equation the index notation is used and the Einstein summation convention holds. In addition, the coma denotes the operation of calculating the partial derivative, e.g.

$$\frac{\partial U_1(X_1, X_2, X_3, t)}{\partial X_1} = U_{1,1}$$

Additionally, the following notation is applied

- U_i – displacement vector component directed along the X_i-axis of the coordinate system at instant t
- G, λ – material constants dependent on the X_i coordinates, $G = G(X_1, X_2, X_3)$ and $\lambda = \lambda(X_1, X_2, X_3)$. These are functions that describe local shear moduli and the Lame constants for a heterogeneous medium
- ρ – mass density dependent on the X_i-coordinates
- γ_i – unit weight component dependent on X_i-coordinates.

Coming on to the 1D issue: $U = U_1$, $U_2 = 0$, $U_3 = 0$, which can be called the oedometric equation of movement, we get .

$$\begin{aligned} &\frac{\partial}{\partial X}\left[E_d(X)\frac{\partial}{\partial X}U\right] + \gamma(X) = \rho(X)\frac{\partial^2}{\partial t^2}U \\ &E_d(X)\frac{\partial^2}{\partial X^2}U + \gamma = \rho\frac{\partial^2}{\partial t^2}U \end{aligned} \tag{2.2}$$

The former equation refers to a heterogeneous body whereas the latter – to a homogeneous medium. These equations emarge from the assumption that $X = X_1$, $U = U_1$, $E_d = \lambda + 2G$ and $\gamma = \gamma_1$. The latter equation is derived from the former on the assumption that the oedometric modulus of compressibility E_d, unit weigh γ, and mass density ρ, do not depend on the spatial coordinate X.

As an example, we shall solve the problem of determining the amplitude of steady-state vibrations $W(X)$ in a heterogeneous sample under the oedometric condition of deformation. The boundary conditions were assumed as follows: $W(0) = A$, $W(L) = 0$ for a sample showing a periodic heterogeneity of the oedometric modulus of elasticity $E_d(X)$ and mass density $\rho(X)$. Sample oscillations of the material constants of an l-period are shown in Fig.1.

The equation representing the problem is the first term of (2.2), after the unit weight effect in it has been neglected

$$\frac{\partial}{\partial X}\left[E_d(X)\frac{\partial}{\partial X}U\right] = \rho(X)\frac{\partial^2}{\partial t^2}U \tag{2.3}$$

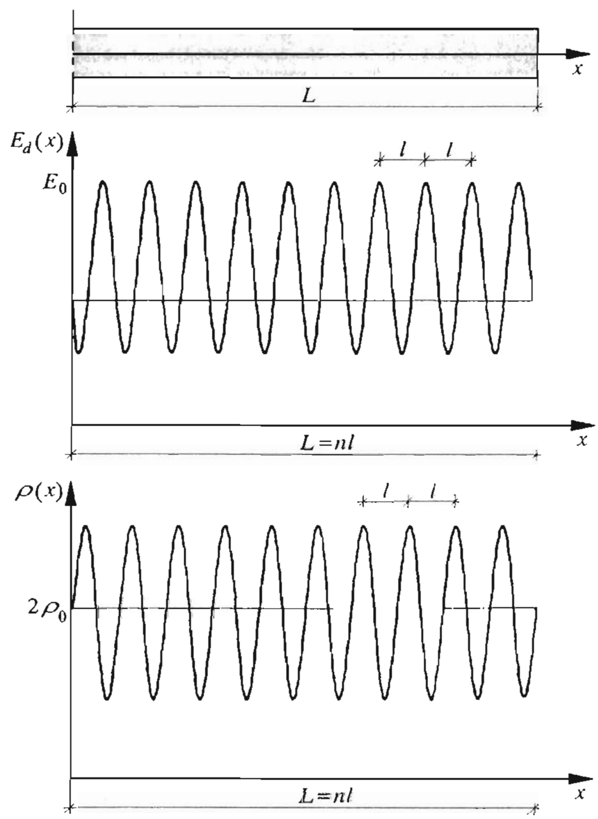

Fig. 1. Assumed variations of modulus E_d and density ρ along the sample axis

The assumption that the sample exhibits stationary vibrations

$$U(X,t) = W(X)\exp(\mathrm{i}\omega t)$$

resolves the above issue to solving the Helmholtz wave equation

$$\frac{\partial}{\partial X}\Big[E_d(X)\frac{\partial}{\partial X}W\Big] + \rho(X)\omega^2 W = 0 \tag{2.4}$$

Unfortunately, the general solution of this equation cannot be written down in the closed form for any periodical functions $E_d(X)$ and $\rho(X)$. The particular form of Eq (2.4) is Hill's equation: $y'' + [f(x) + e]y = 0$, where $f(x)$ is a periodic function. A further simplification may lead to Mathie's equation: $y'' + (b\cos 2x + h)y = 0$. However, these simplified forms of the equation do not have general solutions expressed by elementary functions either. Therefore, in order to find possibly the simplest form solution of Eq (2.4), we assume the particular forms of $E_d(X)$ and $\rho(X)$ periodic changes presented in Fig.1

$$E_d(X) = \frac{E_0}{2+\sin\frac{2\pi X}{l}} \qquad \rho(X) = \rho_0\Big(2+\sin\frac{2\pi X}{l}\Big) \tag{2.5}$$

The general solution of Eq (2.4), having taken into account the above substitutions, the notation: $c = \sqrt{E_0/\rho_0}$ and the boundary conditions $W(0) = A$, $W(L) = 0$ reads

$$W(X) = -\frac{A}{\sin\alpha + \cos\alpha\tan\varphi}\sin\Big(\frac{\omega L}{c}\xi_1\Big) + \frac{A\tan\varphi}{\sin\alpha+\cos\alpha\tan\varphi}\cos\Big(\frac{\omega L}{c}\xi_1\Big) \tag{2.6}$$

with the following notation

$$\xi_1 = 2\frac{X}{L} - \frac{\varepsilon}{2\pi}\cos\frac{2\pi X}{l} \qquad \varphi = \frac{\omega L}{c}\Big(2 - \frac{\varepsilon}{2\pi}\cos\frac{2\pi}{\varepsilon}\Big)$$
$$\alpha = \frac{\omega L\varepsilon}{2\pi c} \qquad \varepsilon = \frac{l}{L} \tag{2.7}$$

This solutions depend on two dimensionless parameters, $\omega L/c$ and l/L. The first one, consisting of the angular velocity of steady-state vibrations ω, the characteristic dimension of the object considered L and a quantity c beeing similiar to the sound speed in the medium, might be either extremely big: $\omega L/c = O(\varepsilon^{-1})$; it can be of the order of one: $\omega L/c = O(1)$, or extremely small: $\omega L/c = O(\varepsilon)$, where the parameter ε is much smaller than one: $\varepsilon \ll 1$. This quotient determines the ratio of sample length to the wave length that can propagate in the sample. The l/L coefficient can assume values of the order of one: $l/L = O(1)$, or very small ones: $l/L = O(\varepsilon)$. Noticeably, the values of the order of one are only possible if the method of direct homogenisation is applied. The solution (2.6), depending on the values of the two quotients, can take six different forms.

Case I – l/L-parameter is of the order of one and $\omega L/c$ takes a very great value

$$\frac{l}{L} = O(1) \qquad \frac{\omega L}{c} = O(\varepsilon^{-1}) \tag{2.8}$$

In the solution of Eq (2.6), as $\omega L/c$ tends to infinity the result depends on the value of the denominator $\sin\alpha + \cos\alpha\tan\varphi$. If this value is equal to zero, the amplitude distribution is indefinite, as the phenomenon of resonance occurs. For great values of $\omega L/c$, already relatively small changes of ω or c might result in such a situation. The solution assumes an especially simple form for $\varphi = \pi/2 + n\pi$. For this φ-value, the solution (2.6) is simplified to yield

$$W(X) = \frac{A}{\cos\alpha}\cos\Big(\frac{\omega L}{c}\xi_1\Big) \qquad \xi_1 = 2\frac{X}{L} - \frac{l}{2\pi L}\cos\frac{2\pi X}{l} \qquad \alpha = \frac{\omega L}{2\pi c} \tag{2.9}$$

Such a type of solution cannot be obtained using methods of periodic media homogenisation because the *epsilon* parameter is of the order of one: $\varepsilon = l/L = O(1)$, and the homogenisation method requires that *epsilon* be of the order: $O(\varepsilon)$. This case is a part of a different asymptotic theory – the theory of the eiconal of macroscale periodic heterogeneity media.

Case II – The parameters, l/L and $\omega L/c$, are of the order of one

$$\frac{l}{L} = O(1) \qquad \frac{\omega L}{c} = O(1) \tag{2.10}$$

It is Eq (2.6) that makes up the solution in this case. It belongs to the class of solutions of macroscale-heterogeneous media dynamic theory of elasticity.

Case III – l/L-parameter is of the order of one and the $\omega L/c$ quotient is very small

$$\frac{l}{L} = O(1) \qquad \frac{\omega L}{c} = O(\varepsilon) \qquad \varepsilon \ll 1 \tag{2.11}$$

This assumption causes the variables $\omega L\xi_1/c$, φ, α in Eq (2.7) to be extremely small

$$\alpha = \frac{\omega L}{2\pi c} = O(\varepsilon) \qquad \varphi = O(\varepsilon) \qquad \frac{\omega L}{c}\xi_1 = O(\varepsilon) \tag{2.12}$$

After the limit passage changes Eq (2.6) can be rewritten as

$$U(X) = -\frac{A}{M}\Big(\frac{2X}{L} - \frac{l}{2\pi L}\cos\frac{2\pi X}{l}\Big) + A\Big(1 - \frac{l}{2\pi ML}\Big)$$
$$M = 2 + \frac{l}{2\pi L}\Big(1 - \cos\frac{2\pi L}{l}\Big) \tag{2.13}$$

Eq (2.13) belongs to the class of static solutions of the theory of elasticity for macroscale-heterogeneous media.

The theory of homogenisation of micro-scale heterogeneous media does not cover the three cases given above as it requires that the quotient l/L should be very small, of the order of *epsilon*: $O(\varepsilon)$. The first and third cases can be obtained within the framework of other asymptotic theories, where another physical value rather than the relative periodicity cell size of the medium heterogeneous material constants is a small parameter.

Case IV – l/L-parameter is very small and the coefficient $\omega L/c$ is of a very great value

$$\frac{l}{L} = O(\varepsilon) \qquad \frac{\omega L}{c} = O(\varepsilon^{-1}) \qquad \varepsilon \ll 1 \tag{2.14}$$

This is a sample non-homogenisable issue in the theory of microscale-heterogeneous media homogenisation. On the assumption (2.14) the variables in Eq (2.7) take the values of the following orders of magnitude

$$\begin{aligned} &\alpha = \frac{\omega L \varepsilon}{2\pi c} = O(1) \qquad \varphi = O(\varepsilon^{-1}) \\ &\frac{\omega L}{c}\xi_1 = \frac{2\omega X}{c} - \frac{\omega L\varepsilon}{2\pi c}\cos\frac{2\pi X}{l} = 2n\pi + \varphi_1 \end{aligned} \tag{2.15}$$

The possible solutions include that of the type of Eq (2.9), which reads

$$W(X) = \frac{A}{\cos\alpha}\cos\Big(\frac{\omega L}{c}\xi_1\Big) \tag{2.16}$$

It belongs to the class of solutions of geometric optics for heterogeneous bodies for which the wavelength c is comparable with the length of the medium heterogeneity periodicity cell.

Case V – The parameter $l/L = \varepsilon$ is very small and the ratio $\omega L/c$ is of the order of one

$$\frac{l}{L} = O(\varepsilon) \qquad \frac{\omega L}{c} = O(1) \qquad \varepsilon \ll 1 \tag{2.17}$$

For these parameter order of magnitude, what we have is a classical case of applying the process of homogenisation to the dynamic theory of microscale-heterogeneous media. The assumed orders of magnitude for the parameters simplify Eq (2.7) as follows

$$\varphi = \frac{2\omega L}{c} = O(1) \qquad \xi_1 = \frac{2X}{L} \qquad \alpha = \frac{\omega L}{2\pi c}\varepsilon = O(\varepsilon) \tag{2.18}$$

After the limit passage Eq (2.6) reads

$$W(X) = -\frac{A\sin\big(\frac{2\omega}{c}X\big)}{\tan\frac{2\omega L}{c}} + A\cos\Big(\frac{2\omega}{c}X\Big) \tag{2.19}$$

This is a solution that determines stationary vibrations in a homogeneous medium. In the formula above, the parameters consist of the effective material constants of the homogeneous medium, which were formed from periodically variable constants of the heterogeneous medium. Unfortunately, as was mentioned earlier, the method of direct homogenisation fails to give a way to determine the effective oedometric modulus, E_{ef}, and effective mass density ρ_{ef} in the mathematical model of the equivalent homogeneous medium

$$E_{ef}\frac{\partial^2 U}{\partial X^2} = \rho_{ef}\frac{\partial^2 U}{\partial t^2} \tag{2.20}$$

Without referring to the general homogenisation procedure, we can only guess that E_{ef} and ρ_{ef} should be determined with the aid of the formulae

$$E_{ef} = \langle E_d^{-1}(y)\rangle^{-1} = \frac{E_0}{2} \qquad \rho_{ef} = \langle \rho(y)\rangle = 2\rho_0$$
$$c_{ef} = \sqrt{\frac{E_{ef}}{\rho_{ef}}} = \frac{c}{2} \tag{2.21}$$

The symbol $\langle\cdot\rangle$ denotes averaging over the periodicity cell area.

These dependencies allow for determination of the effective wave velocity c_{ef}, equal to the sound speed in the equivalent medium.

Case VI – The two values: the parameter $l/L = \varepsilon$ and the coefficient $\omega L/c$ are very small

$$\frac{l}{L} = O(\varepsilon) \qquad \frac{\omega L}{c} = O(\varepsilon) \qquad \varepsilon \ll 1 \tag{2.22}$$

On these assumptions the variables in Eq (2.7) are of the following order of magnitude

$$\varphi = \frac{2\omega L}{c} = O(\varepsilon) \qquad \frac{\omega L}{c}\xi_1 = \frac{2\omega X}{c} = O(\varepsilon) \qquad \alpha = \frac{\omega L}{2\pi c}\varepsilon = O(\varepsilon^2) \tag{2.23}$$

After the limit passage Eq (2.6) simplifies to the form: $W(X) = A - AX/L$. In this case, the process of direct homogenisation leads to the static solution. This result can be obtained using the equivalent model for a homogeneous body, by considering the analogous boundary-value problem.

The six cases of direct homogenisation are summed up in Table 1. The basic problem of dynamics of periodically heterogeneous bodies given in bold can be simplified in five different ways as a result of direct homogenisation.

Table 1. Possible cases of homogenisation of the dynamic problem of periodically heterogeneous media

$\omega L/c$ l/L		$O(\varepsilon^{-\alpha})$	$O(1)$	$O(\varepsilon^{\alpha})$
$O(1)$	Theory Medium Transition	Optics Heterogeneous Asymptotic	**Dynamics** **Heterogeneous**	Statics Heterogeneous Asymptotic
$O(\varepsilon^{-\alpha})$	Theory Medium Transition	Optics Heterogeneous Non-homogenisable	Dynamics Equivalent Homogenisation	Statics Equivalent Homogenisation

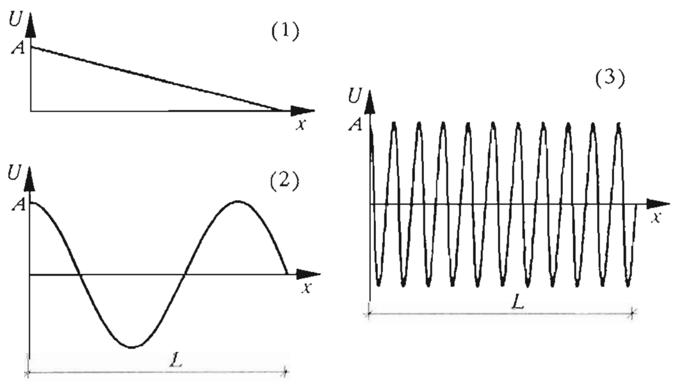

Fig. 2. Solutions yielding the amplitude distribution in the sample given in the first row of Table 1: quasi-static approximation (1), solution of the dynamic theory of elasticity of heterogeneous media (2) and solution of the optical-geometric approximation (3)

Two limit passages for the first row are not included in the homogenisation theory discussed. The theory of homogenisation comprises limit passages up to the second-row cases.

3. Flow of a Newtonian liquid through a non-deformable porous medium

Let us assume that a porous medium constitutes a non-deformable structure formed of a solid. Inside this structure, there is a network of filtration channels, linked mutually regularly enough to determine the Representative Volume Element (RVE), meeting the conditions of structural periodicity. Our assumption is that the considered body contains a big number of such repeatable elements, which can be presented in Fig.3.

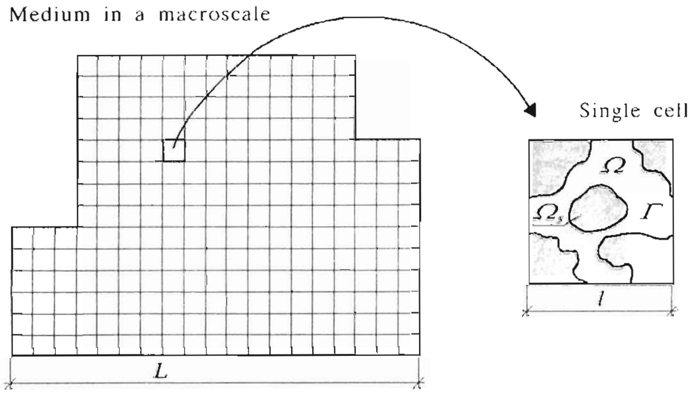

Fig. 3. Cross-section through the periodic structure of porous medium

An incompressible Newtonian liquid flows through the pores and this phenomenon takes place at a constant temperature (isothermal process). The way of derivation of the equations describing the process of liquid flow through a periodic cell was presented in several works, cf Auriault et al. (1990), Mei and Auriault (1989), (1991), Sanchez Palencia (1974), (1980), Strzelecki et al. (1996), Whitaker (1986).

Taking into account the postulate of Newtonian liquid incompressibility, we can present the system of equations, describing the flow process through the pores of a non-deformable medium, in the following form

$$
\begin{gathered}
\mu\nabla^2\boldsymbol{v}^f - \operatorname{grad} p^f = \rho^f\Big(\frac{\partial \boldsymbol{V}^f}{\partial t^f} + \boldsymbol{v}^f \operatorname{grad}\boldsymbol{v}^f\Big) \\
\operatorname{div}\boldsymbol{v}^f = 0 \qquad \boldsymbol{v}^f\Big|_{\Gamma} = \boldsymbol{0} \qquad [\![\boldsymbol{v}^f]\!] = \boldsymbol{0} \qquad [\![p]\!] = 0
\end{gathered}
\tag{3.1}
$$

The above system makes the starting point for homogenisation procedure. However, it must be preceded by an introduction of dimensionless variables; hence the normalisation of this equation system should be performed.

3.1. Normalisation of equations

In equations (3.1), all of the values are physical values with certain units corresponding to the physical sense of these values. Following the algorithm of homogenisation proposed by Beer (1972) we have

$$v_{max} = \max \|\boldsymbol{v}^f\| \qquad p_{\max} = \max \|p^f\|$$
$$\frac{v_{max}}{l^2} = \max \|\nabla^2 \boldsymbol{v}^f\| \qquad \frac{p_{max}}{l} = \max \|\operatorname{grad} p^f\| \tag{3.2}$$
$$\frac{v_{max}}{t_{max}} = \max \|\frac{\partial \boldsymbol{v}^f}{\partial t}\| \qquad \frac{v_{max}^2}{l} = \max \|\boldsymbol{v}^f \operatorname{grad} \boldsymbol{v}^f\|$$

By substituting dependencies (3.2) into equations (3.1), we get

$$\frac{\mu v_{max}}{l^2}\nabla^2\Big(\frac{\boldsymbol{v}^f}{v_{max}}l^2\Big) - \frac{p_{max}}{l}\operatorname{grad}\Big(\frac{p^f}{p_{max}}l\Big) = \frac{\rho^f v_{max}}{t_{max}}\frac{\partial}{\partial t^f}\Big(\frac{\boldsymbol{v}^f}{v_{max}}t_{max}\Big) +$$
$$+\frac{\rho^f v_{max}^2}{l}\Big(\frac{\boldsymbol{v}^f}{v_{max}}\operatorname{grad}\frac{\boldsymbol{v}^f}{v_{max}}\Big) \tag{3.3}$$
$$\operatorname{div}\frac{\boldsymbol{v}^f}{v_{max}} = 0 \qquad \frac{\boldsymbol{v}^f}{v_{max}}\Big|_{\Gamma} = 0$$

Then we shall introduce the following dimensionless value of velocity and pressure

$$\boldsymbol{v} = \frac{\boldsymbol{v}^f}{v_{max}} \qquad p = \frac{p^f}{p_{max}} \tag{3.4}$$

We define all the functions of physical values in a double magnitude scale (X, Y), hence

$$\boldsymbol{v} = \boldsymbol{v}(X, Y) \qquad p = p(X, Y) \tag{3.5}$$

while: $X \in [0, L]$ is the macroscopic spatial (physical) variable, $Y \in [0, l]$ is the local spatial (physical) variable.

In the general case, there is a simple linear relationship between the macroscopic and the local variables, i.e. $X = Y + C$, where C denotes a constant value.

The values are Ω-periodical with respect to the Y-variable, which means that: $\boldsymbol{v}(X, Y + l) = \boldsymbol{v}(X, Y)$ and $p(X, Y + l) = p(X, Y)$. As we tend to resolve our system of equations to the dimensionless form, we shall introduce the dimensionless spatial coordinates x and y, as well as a dimensionless time t

$$x = \frac{X}{L} \qquad y = \frac{Y}{l} \qquad x, y \in [0, 1] \qquad t = \frac{t^f}{t_{max}} \qquad t \in [0, \infty) \tag{3.6}$$

The derivative of a function with respect to the X-macroscopic dimensional variable has the form

$$\frac{d}{dX} = \frac{1}{l}\Big(\varepsilon\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) \qquad \varepsilon = \frac{l}{L} \tag{3.7}$$

Let us now introduce other dimensionless values

$$Q_l = \frac{p_{max}/l}{\mu v_{max}/l^2} \qquad R_{el} = \frac{\rho^f v_{max}^2/l}{\mu v_{max}/l^2} \qquad R_{tl} = \frac{\rho^f v_{max}^2/t_{max}}{\mu v_{max}/l^2} \tag{3.8}$$

The quantities Q_l, R_{el} and R_{tl} have their physical meaning. The dimensionless number Q determines the ratio between the forces due to pressure gradient and the viscous drag in the flowing liquid

$$Q_l = O\Big(\frac{\operatorname{grad} p}{\mu \operatorname{div} \boldsymbol{v}}\Big) \tag{3.9}$$

In Auriault's papers (Auriault, 1980, 1987, 1994; Auriault et al., 1990) it was assessed that, in the case of a liquid flow through a porous medium, the values of Q_l is of the order of $O(\varepsilon^{-1})$.

The following reasoning was made:

- The order of magnitude the pressure gradient observed in the macroscale was determined

$$\operatorname{grad} p = O\Big(\frac{p}{l}\Big) = O\Big(\frac{p}{\varepsilon L}\Big) = O(\varepsilon)$$

  - then the viscous drag was assessed

$$\mu \operatorname{div} \boldsymbol{v} = O\Big(\frac{\mu\|\boldsymbol{v}\|}{l^2}\Big) = O(1)$$

  - the ration between the two values finally yields the order of magnitude

$$Q_l = O(\varepsilon^{-1}) \tag{3.10}$$

Q_l may certainly assume a value much greater or smaller than that assumed in Eq (3.10). We shall perform homogenisation assuming the following orders of magnitude for dimensionless variable Q_l

$$Q_l = O(\varepsilon^0) \qquad Q_l = O(\varepsilon^{-1}) \qquad Q_l = O(\varepsilon^{-2}) \tag{3.11}$$

The first of the above cases means that, in the microscale, the pressure gradient is of the same order of magnitude as the viscous drag. Thus, in the macroscale, we deal with the classical problem of hydraulics. The last of the cases, i.e. when $Q_l = O(\varepsilon^{-2})$ means that the pressure gradient is of the order of $Q_l = O(\varepsilon^{-1})$, when the viscous drag is of the order of $Q_l = O(\varepsilon)$, i.e. the load factor is ε^{-2} stronger than the drag forces. Auriault (1994) suggested that the order of the R_{el}-value was associated with Q_l in the following way

$$R_{el} = O(Q^{-1}) \tag{3.12}$$

which would indicate that the order of magnitude of the two numbers is determined the order of magnitude of dimensionless velocity v. Should this suggestion be taken as true, then we would need to analyse the following combinations of the dimensionless constants orders of magnitude

$$\begin{array}{lcl} Q_l = O(\varepsilon^0) & \text{and} & R_{el} = O(\varepsilon^0) \\ Q_l = O(\varepsilon^{-1}) & \text{and} & R_{el} = O(\varepsilon) \\ Q_l = O(\varepsilon^{-2}) & \text{and} & R_{el} = O(\varepsilon^2) \end{array} \tag{3.13}$$

In the general case, after assuming initially three orders of magnitude of Q_l and, analogously, three orders of R_{el} we would need to consider 9 combinations of mutual relationships between these values. If, in addition, we take into account the term of inertial forces, i.e. three cases of dimensionless Reynolds number, we would have to consider 27 cases with various configurations of the dimensionless values Q_l, R_{el} and R_{tl}.

In our paper we shall confine ourselves to an analyses of a much smaller number of mutual relations of the dimensionless values Q_l, R_{el} and R_{tl}, though such that yield interpretations which are significant from the viewpoint of physics. In the case of a steady flow, we shall confine ourselves to the cases described by Eqs (3.13). Taking into account Eqs (3.4), (3.6) $\div$ (3.8), in the system of equations (3.3) we get the normalised system of equations in the form

$$\begin{gathered} [\varepsilon^2\nabla_x^2 + 2\varepsilon\nabla_{xy}^2 + \nabla_y^2]\boldsymbol{v} - Q_l[\varepsilon\,\mathrm{grad}_x + \mathrm{grad}_y]p = R_{tl}\frac{\partial \boldsymbol{v}}{\partial t} + \\ +R_{el}[\varepsilon\boldsymbol{v}\,\mathrm{grad}_x + \boldsymbol{v}\,\mathrm{grad}_y]\boldsymbol{v} \\ \varepsilon\,\mathrm{div}_x\boldsymbol{v} + \mathrm{div}_y\boldsymbol{v} = 0 \qquad \boldsymbol{v}\Big|_\Gamma = \mathbf{0} \end{gathered} \tag{3.14}$$

The x,y-indices at the symbols grad, div, ∇^2 mean that dimensionless variables x, y are the independent variables with rspect to which differentiation is performed. The above system of equations is supplemented by the conditions of Ω-periodicity for the functions of pressure p, and velocity $\boldsymbol{v}$

$$[\![p]\!] = 0 \qquad [\![\boldsymbol{v}]\!] = 0 \tag{3.15}$$

All of the vector and scalar values in Eqs (3.14) lie within the range $[0,1]$ except for time t, which changes within the range $[0,\infty)$.

Then we shall confine ourselves to considerations of a steady liquid flow. In this case, the system of equations (3.14) is simplified to the form

$$[\varepsilon^2\nabla_x^2 + 2\varepsilon\nabla_{xy}^2 + \nabla_y^2]\boldsymbol{v} - Q_l[\varepsilon\,\mathrm{grad}_x + \mathrm{grad}_y]p =$$

$$= R_{el}[\varepsilon\boldsymbol{v}\,\mathrm{grad}_x + \boldsymbol{v}\,\mathrm{grad}_y]\boldsymbol{v} \tag{3.16}$$

$$\varepsilon\,\mathrm{div}_x\boldsymbol{v} + \mathrm{div}_y\boldsymbol{v} = 0 \qquad \boldsymbol{v}\Big|_\Gamma = \boldsymbol{0}$$

with the periodicity conditions (3.15).

The homogenisation procedure proposes expanding the sought functions $\boldsymbol{v}$ and p into an asymptotic series with respect to the small parameter ε

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}^{(0)} + \varepsilon\boldsymbol{v}^{(1)} + \varepsilon^2\boldsymbol{v}^{(2)} + \varepsilon^3\boldsymbol{v}^{(3)} + ... \\ p &= p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \varepsilon^3 p^{(3)} + ... \end{aligned} \tag{3.17}$$

The process of normalisation allowed us to determine dimensionless system of equation (3.16), which can now be subjected to the homogenisation procedures.

3.2. Homogenisation

Let us start with the case of dimensionless values Q_l i R_{el} being of the order

$$Q_l = O(\varepsilon^{-1}) \qquad R_{el} = O(\varepsilon) \tag{3.18}$$

Taking into account Eqs (3.16) with the periodicity conditions (3.15) we have

$$[\varepsilon^2\nabla_x^2 + 2\varepsilon\nabla_{xy}^2 + \nabla_y^2]\boldsymbol{v} - \varepsilon^{-1}[\varepsilon\,\mathrm{grad}_x + \mathrm{grad}_y]p =$$

$$= \varepsilon[\varepsilon\boldsymbol{v}\,\mathrm{grad}_x + \boldsymbol{v}\,\mathrm{grad}_y]\boldsymbol{v} \tag{3.19}$$

$$\varepsilon\,\mathrm{div}_x\boldsymbol{v} + \mathrm{div}_y\boldsymbol{v} = 0 \qquad \boldsymbol{v}\Big|_\Gamma = \boldsymbol{0} \qquad [\![\boldsymbol{v}]\!] = \boldsymbol{0} \qquad [\![p]\!] = 0$$

After substituting the asymptotic expansions (3.17) into Eqs (3.19), we obtain an infinite system of differential equations corresponding to the subsequent powers of ε^i. For the lowest orders of the expansion with respect to ε, we get the following systems of equations:

— for the order of magnitude ε^{-1}

$$\mathrm{grad}\,p^{(0)} = \boldsymbol{0} \tag{3.20}$$

— for the order of magnitude $\varepsilon^{(0)}$

$$\operatorname{div}_y \boldsymbol{v}^{(0)} = 0 \qquad \boldsymbol{v}^{(0)}\Big|_{\Gamma} = \mathbf{0} \qquad [\![\boldsymbol{v}^{(0)}]\!] = \mathbf{0} \qquad [\![p^{(0)}]\!] = 0$$
$$\nabla_y^2 \boldsymbol{v}^{(0)} - \operatorname{grad}_y p^{(1)} - \operatorname{grad}_x p^{(0)} = \mathbf{0} \tag{3.21}$$

— for the order of magnitude ε^1

$$\operatorname{div}_y \boldsymbol{v}^{(1)} + \operatorname{div}_x \boldsymbol{v}^{(1)} = 0 \qquad \boldsymbol{v}^{(1)}\Big|_{\Gamma} = \mathbf{0} \qquad [\![\boldsymbol{v}^{(1)}]\!] = \mathbf{0} \qquad [\![p^{(1)}]\!] = 0$$
$$\nabla_y^2 \boldsymbol{v}^{(1)} + 2\nabla_{xy}^2 \boldsymbol{v}^{(0)} - \operatorname{grad}_y p^{(2)} - \operatorname{grad}_x p^{(1)} = \boldsymbol{v}^{(0)} \operatorname{grad}_y \boldsymbol{v}^{(0)} \tag{3.22}$$

— for the order of magnitude ε^2

$$\operatorname{div}_y \boldsymbol{v}^{(2)} + \operatorname{div}_x \boldsymbol{v}^{(1)} = 0 \qquad \boldsymbol{v}^{(2)}\Big|_{\Gamma} = \mathbf{0} \qquad [\![\boldsymbol{v}^{(2)}]\!] = \mathbf{0} \qquad [\![p^{(2)}]\!] = 0 \tag{3.23}$$

...

However, having considered only the first two equations, we can already reach the expected results referring to the mathematical model of the equivalent medium and the way of averaging the material constants of the macroscopic medium. After taking into account the periodicity condition Eq (3.20) implies that the value of pressure $p^{(0)}$ depends solely on macroscopic variable x, i.e. in the local scale y is constant

$$p^{(0)} = p^{(0)}(x) \tag{3.24}$$

In order to obtain the relationship between the values of highest order asymptotic expansions (3.17), the task resolves itself to solving the system of equations (3.21) ÷ (3.23), which yields

$$\nabla_y^2 \boldsymbol{v}^{(0)} - \operatorname{grad}_y p^{(1)} - \operatorname{grad}_x p^{(0)} = \boldsymbol{v}^{(0)}$$
$$\operatorname{div}_y \boldsymbol{v}^{(0)} = 0 \qquad \boldsymbol{v}^{(0)}\Big|_{\Gamma} = \mathbf{0} \qquad [\![\boldsymbol{v}^{(0)}]\!] = \mathbf{0} \qquad [\![p^{(1)}]\!] = 0 \tag{3.25}$$

The following make up the solution of the above system

$$\boldsymbol{v}^{(0)} = -\mathbf{K}(y) \operatorname{grad}_x p^{(0)}$$
$$p^{(1)} = \boldsymbol{\tau}(y) \operatorname{grad}_x p^{(0)} + \tilde{p}^{(1)}(x) \tag{3.26}$$

where the tensor $\mathbf{K}(y)$ and vector $\boldsymbol{\tau}(y)$ are functions of the local variable, while $\tilde{p}^{(1)}(x)$ is a function depending on the macroscopic variable x.

Substituting Eq (3.26) into Eqs (3.25), we can see that functions $\mathbf{K}(y)$ and $\boldsymbol{\tau}(y)$ must fulfil the system of equations

$$\nabla_y^2\mathbf{K}(y) + \frac{\partial\boldsymbol{\tau}(y)}{\partial y_j} + \delta_{ij} = 0$$
$$\operatorname{div}_y\mathbf{K}(y)\mathbf{I} = 0 \qquad \mathbf{K}(y)\Big|_{\Gamma} = \mathbf{0} \qquad [\![\mathbf{K}(y)]\!] = \mathbf{0} \qquad [\![\boldsymbol{\tau}(y)]\!] = \mathbf{0} \tag{3.27}$$

After being averaged, Eqs (3.27) constitute the starting point for determining the value of Darcy's macroscopic coefficient of filtration. If numerical methods are applied, Eqs (3.27) make the starting point to formulate the boundary-value problem. The obtained functions $\mathbf{K}(y)$ and $\boldsymbol{\tau}(y)$, after averaging, render it possible to find the second-order tensor $\mathbf{K}$ and vector $\boldsymbol{\tau}$.

Rewriting the first solution of Eq (3.25) intoterms of the filtration velocity vector components, we have

$$v_i^{(0)} = -k_{ij}(y)\frac{\partial p^{(0)}}{\partial x_j} \tag{3.28}$$

Passing on to the physical variables in Eq (3.28) as well as multiplying and dividing the right-hand side of the equation by μl, we get

$$v_i^{f(0)} = -\frac{k_{ij}(y)}{\mu}\frac{\mu v_{max}}{p_{max}l}l\,L\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.29}$$

hence

$$v_i^{f(0)} = -\frac{k_{ij}(y)}{\mu}Q_l^{-1}\varepsilon^{-1}l^2\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.30}$$

As we assumed that $Q_l = O(\varepsilon^{-1})$, we can then write down the following on the assumption that, in a particular case, the value of constant $A = O(1) = 1$

$$Q_l^{-1} = A\varepsilon = \varepsilon \tag{3.31}$$

Substituting Eq (3.31) into Eq (3.30), we get the final solution of the system of equations in the form

$$v_i^{f(0)} = -k_{ij}(y)\frac{l^2}{\mu}\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.32}$$

After averaging relative to spatial coordinate y, we have

$$\langle v_i^{f(0)} \rangle = -\frac{l^2}{\mu} \langle k_{ij}(y) \rangle \frac{\partial p^{f(0)}}{\partial X_j} \tag{3.33}$$

where

$$\langle (\cdot) \rangle = \frac{1}{|\Omega|} \int_{\Omega} (\cdot) \, dv \tag{3.34}$$

Eq (3.33) is Darcy's filtration law, known from experimental physics. This law can be written down in the form

$$\langle v_i^{f(0)} \rangle = -\tilde{k}_{ij} \frac{\partial p^{f(0)}}{\partial X_j} \tag{3.35}$$

where $\tilde{k}_{ij}$ is the second-order tensor of permeability, the numeric value of which is of the order

$$\tilde{k}_{ij} = O\Big(\frac{l^2}{\mu}\Big) \tag{3.36}$$

whereas k_{ij} is a mean value $\langle k_{ij}(y) \rangle$ dependent on the cell internal structure considered on the microscale.

Let us consider the physical sense of all the values obtained by averaging in Eq (3.33). What rises suspicion is the sense of mean value $\boldsymbol{v}^{f(0)}$, defined as a value of volumetric mean, while filtration velocity is associated with a flow through the surface, so we should calculate the mean over the surface. In reality, we can show that, in this case, the two means are equal. This is due to the selenoidal character of function $\boldsymbol{v}^{(0)}$.

Now we shall consider the case when the dimensionless values Q_l and R_l are of the order of magnitude

$$Q_l = O(1) \qquad R_{el} = O(1) \tag{3.37}$$

Physically, this means that the pressure gradient is of the same order as the viscous drag and the forces of convection. Taking into account Eq $(3.37)_1$ in the system of equations (3.16) with periodicity condition (3.15) and then applying asymptotic expansion of the form (3.17), after analogous mathematical transformation, as in the previous case, we have

$$\boldsymbol{v}^{(0)} = \mathbf{0} \qquad p^{(0)} = p^{(0)}(x) \tag{3.38}$$

Hence, in case we are considering the ideal homogenisation, i.e. such that *epsilon* is very close to zero, we do not observe any flow of the liquid even though there appears some variation of pressure $p^{(0)}(x)$ on the macroscale. For the term of order ε^1 we get

$$\begin{aligned} \boldsymbol{v}^{(1)} &= -\mathsf{K}(y)\,\mathrm{grad}_x p^{(0)} \\ p^{(1)} &= \boldsymbol{a}(y)\,\mathrm{grad}_x p^{(0)} + \tilde{p}^{(1)}(x) \end{aligned} \tag{3.39}$$

where $\mathsf{K}(y)$ and $\boldsymbol{a}(y)$ are functions of local variable y, and $\tilde{p}^{(1)}(x)$ is a function dependent solely on the macroscopic variable x.

Functions $\mathsf{K}(y)$ and $\boldsymbol{a}(y)$ are the solutions of the following equations system

$$\begin{aligned} &\nabla_y^2\mathsf{K}(y) + \frac{\partial a_i(y)}{\partial y_j} + \delta_{ij} = 0 \\ &\mathrm{div}_y\mathsf{K}(y)\mathsf{I} = 0 \qquad \mathsf{K}(y)\Big|_{\Gamma} = \mathbf{0} \qquad [\![\mathsf{K}(y)]\!] = \mathbf{0} \qquad [\![\boldsymbol{a}(y)]\!] = \mathbf{0} \end{aligned} \tag{3.40}$$

Writing the first solution of Eq (3.39) in terms of the velocity vector components, we get

$$v_i^{(0)} = -k_{ij}(y)\frac{\partial p^{(0)}}{\partial x_j} \tag{3.41}$$

Passing to dimensional variables in equation (3.41), we shall get

$$v_i^{f(0)} = -\frac{k_{ij}(y)}{\mu}l^2\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.42}$$

If we take the velocities from the asymptotic expansion of Eq (3.17) with an accuracy of small of the first order, it is expressed by the formula

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \varepsilon\boldsymbol{v}^{(1)}$$

hence

$$\varepsilon^{-1}v_i^f = -\frac{k_{ij}(y)}{\mu}l^2\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.43}$$

By averaging relative to the local coordinate y, we get

$$\langle v_i^f\rangle = -\varepsilon\tilde{k}_{ij}\frac{\partial p^{f(0)}}{\partial X_j} \tag{3.44}$$

The above formula clearly implies that for a very small value of ε the mean velocity of filtration tends to zero: $\langle v_i^f\rangle = 0$. Following this, the filtration velocity is, for $\varepsilon \to 0$, equal to zero although pressure gradient $p^{(0)}$ might not be equal zero. This case describes filtration of an extremely viscous liquid, forced by the limited value of pore pressure gradient $\partial p^{(0)}/\partial X_i = O(1)$. Only an infinitely high pressure gradient, $\partial p^{(0)}/\partial X_i = O(\varepsilon^{-1})$, not occurring in practice, makes a noticeable liquid flow. Let us consider the last case, i.e. when

$$Q_l = O(\varepsilon^{-2}) \qquad \text{and} \qquad R_{el} = O(\varepsilon^2) \tag{3.45}$$

By analogous procedures as in the previous cases we get $p^{(0)} = \text{const}$, and $p^{(1)}$ is a function of only one macroscopic variable, i.e.

$$p^{(1)} = p^{(1)}(x) \qquad \text{and} \qquad v_i^{(0)} = -k_{ij}(y)\frac{\partial p^{(0)}}{\partial x_j} \tag{3.46}$$

After averaging, we obtain Darcy's law again, yet with the excitation smaller by an order than in the first case under consideration. By introducing the first two terms from the pressure asymptotic expansion

$$p = p^{(0)} + \varepsilon p^{(1)} \tag{3.47}$$

and passing on to dimensional variable in Eq $(3.46)_2$, we get

$$v_i^{(0)f} = -k_{ij}(y)\frac{l^2}{\mu}\varepsilon^{-1}\frac{\partial p^f}{\partial X_j}$$

Thence, after averaging, we have

$$\langle v_i^{(0)f}\rangle = -\tilde{k}_{ij}\varepsilon^{-1}\frac{\partial p^f}{\partial X_j} \tag{3.48}$$

Noticeably, $\partial p/\partial X_j$ must be of the order of $O(\varepsilon)$, so that $\langle v_i^{(0)}\rangle$ can be of the order of $O(1)$, which means that this solution is reasonable only when we deal with very small excitations. For instance, it should be understood physically that we are considering the problem of filtration of a nearly ideal liquid, of minimal kinematic viscosity, of the order of $O(\varepsilon)$. Only very small pressure gradient ensures filtration velocity displaying values of the order of $O(1)$. Visibly, the obtained solution imposes conditions for pressure gradient values on the macroscale. The process is homogenisable.

3.3. The essence of the solutions obtained

We have considered three cases of the homogenisation process, having assumed, respectively, different dimensionless values $Q_l = O(1)$, $Q_l = O(\varepsilon^{-1})$

and $Q_l = O(\varepsilon^{-2})$, in addition assuming that, due to the velocity order of magnitude, there exists the relationship between Q_l and R_{el}

$$R_{el} = O(Q_l^{-1}) \tag{3.49}$$

In the first considered case we have Darcy's filtration law on the macroscale (3.35)

$$\langle v_i^{(0)} \rangle = -\tilde{k}_{ij} \frac{\partial p^{(0)}}{\partial X_j} \tag{3.50}$$

whereas on the microscale the classical description of a incompressible liquid flow (the Navier-Stokes equation) is in force, while $\tilde{k}_{ij}$ is the second-order tensor of permeability and has the order of magnitude

$$\tilde{k}_{ij} = O\Big(\frac{l^2}{\mu}\Big) \tag{3.51}$$

The solution obtained is a significant achievement of theoretical physics. Using only mathematical tools, when passing on from the microscale to the macroscale, we have obtained an entirely different type of the equation that describe the flow process. We have also managed to determine the order of magnitude for the permeability tensor elements of the equivalent medium. It should be stressed that the obtained results agree with the results of experimental physics and the dependence of the permeability coefficient on the l^2/μ-ratio conforms to the experiments. In addition, it should be stressed that the first person who got the linear relationship between the mean velocity of a liquid flowing through a small diameter pipe and pressure gradient was Poisseuille. He showed that the coefficient in his name's equation is proportional to the squared pipe diameter and inversely proportional to the viscosity of the liquid flowing through the pipe.

Now we shall analyse the solution for the case of the equality

$$Q_l = O(1) \qquad\qquad R_{el} = O(1) \tag{3.52}$$

While observing the process of homogenisation, we shall notice that already for the first value order of $\varepsilon^{(0)}$ we get a system of classical hydraulics equations on the macroscale with periodicity conditions. It is the existence of these conditions that leads to solutions in the form

$$\boldsymbol{v}^{(0)} = \mathbf{0} \qquad\qquad p^{(0)} = p^{(0)}(x) \tag{3.53}$$

Further considerations lead to a Darcy's law-type relationship between the second term of $\boldsymbol{v}^{(1)}$-asymptotic expansion and the pressure gradient $p^{(0)}$ in the form equivalent to Eq (3.44)

$$\langle v_i^{f(1)} \rangle = -\tilde{k}_{ij} \frac{\partial p^{(0)}}{\partial X_j} \tag{3.54}$$

Due to the relationship: $\langle v_i^f \rangle = \varepsilon \langle v_i^{f(1)} \rangle$, in the case of ideal homogenisation ($\varepsilon \to 0$) the above equation allows for drawing the conclusion that, at gradient $p^{(0)}$ of the order of $O(1)$, the filtration velocity v^f might tend to zero. This is a trivial solution to the problem of liquid filtration. Such a case rarely occurs in practice. The assumption leads to a result which is uninteresting from the practical viewpoint. The question is when such a case might occur. The answer to this question is simple. Certainly, one can imagine an issue of dealing with an extremely dense liquid, of very high viscosity, or a pressure gradient, small enough for average viscosity, such that the order of viscous drag is the same as the order of the flow excitation. Then this task resolves itself to the case of incompressible liquid filtration with the viscosity coefficient tending to infinity. The process of homogenisation and intuition prompt that the only solution that determines the filtration velocity is a trivial solution, i.e. no liquid flow. In the extreme case of infinite liquid viscosity coefficient, the two-phase medium transforms itself into a single-phase non-deformable material. What we are left with is the third case, when the dimensionless values are as follows

$$Q_l = O(\varepsilon^{-2}) \qquad \text{and} \qquad R_{el} = O(\varepsilon^2) \tag{3.55}$$

(3.55) The interpretation of the case is also easy. The accepted assumptions lead to the conclusion that the value of $p^{(0)}$ is, in this case, a constant value, so the gradient $p^{(0)}$ does not occur. We get the way to homogenise a Darcy's-type equation, which binds the value $\langle v_i^{(0)} \rangle$ with gradient $p^{(1)}$ in the form of relationship

$$\langle v_i^{(0)} \rangle = -\tilde{k}_{ij} \varepsilon^{-1} \frac{\partial p^{(1)}}{\partial X_j} \tag{3.56}$$

The above result makes sense only when the pressure gradient is very small, of the order of $O(\varepsilon)$. When the pressure gradient is of the order of $O^{(1)}$, then the velocity $v_i^{(0)}$ tends to infinity. Such a case might occur in the flow of an almost inviscid liquid through a porous medium, at a very small magnitude of excitation due to the pore pressure gradient. Taking this into account, we can imagine that, in the extreme case, the pores are filled with the Pascal ideal inviscid liquid.

In the case of ideal homogenisation, at ε approaching zero, the two processes in question yield the equivalent media which are uninteresting from the practical view-point. In the first case, this is a non-deformable body and in the second – the Pascal liquid. The processes of averaging from the point of view of the homogenisation theory have limitations, hence the issues under consideration are homogenisable though the conclusions stemming from launching these processes impose additional conditions on the value of macroscopic pressure. In the former, the pore pressure gradient should tend to infinity and in the latter – zero. The obtained results are summed up schematically in Table 2 below.

Table 2. Homogenisation results for the problem of a non-compressible liquid steady flow through a stiff periodically heterogeneous medium

Dimensionless value Q_l	$O(1)$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-2})$
Dimensionless value R_{el}	$O(1)$	$O(\varepsilon)$	$O(\varepsilon^{2})$
Theory	None	Darcy's filtration	None
Medium	Equivalent, Non-deformable	Equivalent, Darcy liquid	Equivalent, Pascal liquid
Transition	Homogenisable	Homogenisable	Homogenisable

It should be emphasised that the above table was prepared taking into account the initial assumption $R_{el} = Q_l^{-1}$, which not always has to be satisfied. Then our considerations should be extended to cover all of the combinations of the mutual relationships between the dimensionless numbers.

4. Conclusions

Using two examples, we have shown the significant impact of the order of magnitude of dimensionless geometrical and loading parameters on the physical equations. The equations describe the courses of phenomena and medium type obtained in the procedure of transition from the microscale to the macroscale, provided such a transition is possible. The analysis we have presented also renders it possible to define the range of issues that are homogenisable. The fact of impossibility to perform this procedure does not always mean that the process cannot be described, but the task should be solved as for a heterogeneous medium. In an easily noticeable way, this is illustrated by the directed homogenisation examples we adduced even though the presented examples fail to comprehensively explain the entire problem.

Three examples of different values of dimensionless Q and R_{el} have been shown for filtration of a non-comprehensible Newtonian liquid. The analysis we have performed allowed us to determine the form of the flow equation on the macroscale when the medium pores are filled with a Pascal liquid or a liquid of infinite viscosity (non-deformable liquid). Amongst the three cases under consideration, only the one that allows for determination of a Darcy-type equation in the macroscale. Although, on the microscale, this process is described by a system of differential equations, it seems to be of great practical significance. However, from the theoretical point of view it seems important that when the pores are filled with a Pascal liquid, one cannot arrive at Darcy's equation on the macroscale. This follows the fact that, in numerous publications referring to the theory of consolidation, it was assumed incorrectly that the Darcy-Biot equation describes the process of a Pascal-type non-viscous liquid flow (Derski, 1978; Biot, 1941).

In the case of liquid filtration, the analysis of the effect exerted by dimensionless values was the subject of studies when the flow velocity grows and the laminar movement transforms itself into the turbulent movement on the microscale. In Mei and Auriault's papers (Mei and Auriault, 1989, 1991), the non-linear equation of the filtration flow was obtained on the macroscale in the form of an asymptotic series. It was also shown that, under any circumstances, even powers cannot occur in this expansion, which excludes Chezy's equation, commonly applied to the description of this process.

The methodology we have presented shows the integrity of certain classes of physical processes. The dissimilarity of process descriptions in, for instance optics or dynamics results from neglecting certain effects when the observation method or the scale is changed. According to the authors, this is what the homogenisation theory significance consists of. Owing to it, the description of certain processes on the atomic scale has been successfully associated with their influence on the processes in continuous media mechanics (Bensoussan, 1978).

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Wpływ rzędu wielkości parametrów bezwymiarowych na wyniki procesu homogenizacji

Streszczenie

Wpływ rzędu wielkości parametrów bezwymiarowych na wyniki procedury homogenizacji został pokazany na dwóch przykładach, z których pierwszy dotyczy homogenizacji bezpośredniej, a drugi używa metodologii klasycznej teorii homogenizacji. W pierwszym przykładzie zilustrowano wpływ rzędu wielkości dwóch parametrów bezwymiarowych na rozwiązanie opisujące amplitudę drgań ustalonych. W drugim przykładzie, dotyczącym przepływu nieściśliwej cieczy lepkiej przez ośrodek porowaty, pokazano wpływ rzędu wielkości dwóch parametrów Q i R_{el} na wynikowe równanie konstytutywne. Przeprowadzona analiza dowodzi, że wiele różnych procesów fizycznych może być potraktowanych w sposób podobny, jeżeli poddajemy je homogenizacji.

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