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# NON-LINEAR STABILITY PROBLEM OF SPHERICAL SHELL LOADED WITH TORQUE

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A thin-walled spherical shell is pivoted at both edges. One of the edges may rotate around the shell axis. Moreover, it is loaded with a torque. The problem of shell stability is considered. The system of equations characterizing the problem consists of a non-linear equation of equilibrium and non-linear compatibility equation. Both equations are solved with Bubnov-Galerkin's method, assuming beforehand the form of deflection and force-functions. As a result of the solution, an algebraic equation is obtained, with respect to a dimensionless load parameter. The critical load parameter corresponding to the minimal critical load value is determined from this equation. The number m at which the load parameter has the minimum value determines the mode of stability loss. The paper is supplied with a numerical example.

*Key words:* shells, non-linear stability

## 1. Introduction

A thin-walled spherical shell, being a subject of the analysis, is shown in Figure 1. Its bottom edge is fixed and pivoted. The upper edge is also pivoted, but may rotate around the vertical axis of the shell. The upper edge is loaded with a rotational moment. The problem of the loss of stability is considered. In order to solve the problem, non-linear stability equations are applied, defined in the monograph by Mushtari and Galimov (1957). The system of equations is solved with making use of Bubnov-Galerkin's method with deflection functions and force functions assumed in advance. The final goal is the determination of the critical load. As the problem is very complex, only obtaining a numerical solution is possible.

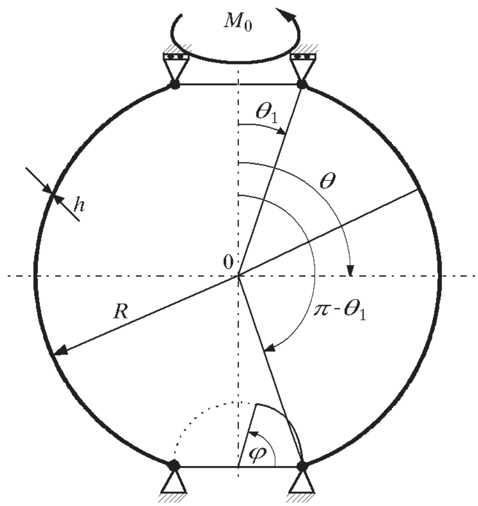


Fig. 1.

## 2. Equations of the stability problem

The system of stability equations takes the following form

$$\begin{aligned}
&\nabla^2\nabla^2\Psi = Eh(\kappa_{12}^2 - \kappa_{11}\kappa_{22} - \kappa_{11}k_{22} - \kappa_{22}k_{11})\\
&D\nabla^2\nabla^2 w + 2\overline{S}\kappa_{12} + T_1(k_{11} + \kappa_{11}) + 2S\kappa_{12} + T_2(k_{22} + \kappa_{22}) = 0
\end{aligned} \tag{2.1}$$

where

| | | |
|---|---|---|
| $\Psi$ | – | force-function |
| $w$ | – | deflection function upon the loss stability |
| $k_{ii}$ | – | main curvatures |
| $\kappa_{ii}, \kappa_{12}$ | – | curvature variations and surface torsion |
| $D$ | – | plate stiffness, $D = Eh^3/[12(1-\nu^2)]$ |
| $\overline{S}$ | – | tangent force in pre-critical state |
| $T_i, S$ | – | critical state forces, and |

$$\nabla^2 = \frac{1}{R^2}\Big(\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Big)$$

Changes in the curvatures and torsions of the spherical shell surface are the following functions of the deflection

$$
\begin{aligned}
&\kappa_{11} = -\frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2} \qquad\qquad \kappa_{22} = -\frac{1}{R^2}\Big(\frac{1}{\sin^2\theta}\frac{\partial^2 w}{\partial\varphi^2} + \cot\theta\frac{\partial w}{\partial\theta}\Big)\\
&\kappa_{12} = \frac{1}{R^2\sin\theta}\Big(\cot\theta\frac{\partial w}{\partial\varphi} - \frac{\partial^2 w}{\partial\varphi\partial\theta}\Big)
\end{aligned}
\tag{2.2}
$$

The forces corresponding to the critical state depend on the force functions as in the following

$$
\begin{aligned}
&T_1 = \frac{1}{R^2}\Big(\frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} + \cot\theta\frac{\partial\Psi}{\partial\theta}\Big) \qquad\qquad T_2 = \frac{1}{R^2}\frac{\partial^2\Psi}{\partial\theta^2}\\
&S = \frac{1}{R^2\sin\theta}\Big(\cot\theta\frac{\partial\Psi}{\partial\varphi} - \frac{\partial^2\Psi}{\partial\varphi\partial\theta}\Big)
\end{aligned}
\tag{2.3}
$$

On the other hand, the tangent force of the pre-critical state takes the form, see Łukasiewicz (1976)

$$
\overline{S} = \frac{M_0}{2\pi R^2\sin^2\theta} \tag{2.4}
$$

Changes in curvatures (2.2) and cross-section forces (2.3), (2.4) should be introduced into equations (2.1). This gives a system of nonlinear partial differential equations with respect to $w$ and $\Psi$.

## 3. Solution to the stability equations

The stability equations are solved with the help of Bubnov-Galerkin's method. This causes the need to adopt a form of the deflection function and the force-functions that possibly satisfies all the boundary conditions of the problem. The edges of the shell are characterized by the following boundary conditions

$$
\begin{aligned}
&\theta = \theta_1 \qquad\qquad w = 0 \qquad\qquad M_\theta = 0\\
&S = \frac{M_0}{2\pi R^2\sin^2\theta_1} \qquad\qquad T_1 = 0
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&\theta = \pi - \theta_1 \qquad\qquad w = 0 \qquad\qquad M_\theta = 0\\
&S = \frac{M_0}{2\pi R^2\sin^2\theta_1} \qquad\qquad T_1 = 0
\end{aligned}
\tag{3.2}
$$

The force functions and deflection functions are assumed in the form

$$\begin{aligned} \Psi &= [b\varphi + c\sin(m\varphi)]\sin^2\theta \\ w &= a\sin\frac{\pi(\theta-\theta_1)}{\pi-2\theta_1}\sin\Big[\frac{\pi(\theta-\theta_1)}{\pi-2\theta_1} + m\varphi\Big]\sin^2\theta \end{aligned} \tag{3.3}$$

where $a, b, c$ are constants and $m$ – an integer number.

Deflection function $(3.3)_2$ meets accurately the first of conditions (3.1), while the second one is met in an integral sense. The force function satisfies the third conditions at both edges, with the accuracy to a constant, while the condition for force $T_1$ remains unsatisfied.

Inseparability equation $(2.1)_1$ is solved with Bubnov-Galerkin's method. The function subject to the orthogonalization has of the form

$$F(\theta,\varphi) = \nabla^2\nabla^2\Psi - Eh(\kappa_{12}^2 - \kappa_{11}\kappa_{22} - \kappa_{11}k_{22} - \kappa_{22}k_{11})$$

while the orthogonalization conditions are

$$\int\limits_A F(\theta,\varphi)f_i(\theta,\varphi)\,dA = 0 \tag{3.4}$$

where

| | | |
|---|---|---|
| $f_i(\theta,\varphi)$ | – | orthogonalization factors, i.e. the components of the $\Psi$ function |
| $A$ | – | middle surface of the shell. |

The orthogonalization conditions are as follows

$$\begin{aligned} &\int\limits_{\theta_1}^{\pi-\theta_1}\int\limits_0^{2\pi} F(\theta,\varphi)\varphi\sin^3\theta\,d\theta d\varphi = 0 \\ &\int\limits_{\theta_1}^{\pi-\theta_1}\int\limits_0^{2\pi} F(\theta,\varphi)\Big)\sin(m\varphi)\sin^3\theta\,d\theta d\varphi = 0 \end{aligned} \tag{3.5}$$

An expansion of conditions (3.5) provides two algebraic equations, serving as a basis for the determination of factors of the $\Psi$ force function. They are presented by the expressions

$$b = EhH_1a^2 \qquad\qquad c = EhH_2a^2 \tag{3.6}$$

where: $H_1$, $H_2$ are the constants including $\theta_1$ and the number $m$.

Finally, the solution to equation $(2.1)_1$ takes the form

$$\varPsi = Eha^2[H_1\varphi + H_2\sin(m\varphi)]\sin^2\theta \tag{3.7}$$

In order to apply the Bubnov-Galerkin method to equilibriu equation $(2.1)_2$ its left-hand side should be considered as the function subject to orthogonalization, with the orthogonalization condition duly defined. The function subject to the orthogonalization is

$$G(\theta,\varphi) = D\nabla^2\nabla^2 w + 2\overline{S}\kappa_{12} + T_1(k_{11} + \kappa_{11}) + 2S\kappa_{12} + T_2\Big(k_{22} + \kappa_{22})$$

while the orthogonalization condition has the form

$$\int\limits_A G(\theta,\varphi)g(\theta,\varphi)\ dA = 0$$

with $g(\theta,\varphi)$ being the orthogonalization factor corresponding to the right-hand side of deflection function $(3.3)_2$. Finally, the orthogonalization condition takes the following form

$$\int\limits_{\theta_1}^{\pi-\theta_1}\int\limits_0^{2\pi} G(\theta,\varphi)\sin\frac{\pi(\theta-\theta_1)}{\pi-2\theta_1}\sin\Big[\frac{\pi(\theta-\theta_1)}{\pi-2\theta_1} + m\varphi\Big]\sin^3\theta\ d\theta d\varphi = 0 \tag{3.8}$$

The mathematical program Derive was used to solve orthogonalization conditions (3.5) and (3.8). Differentiation and integration procedures, and also procedures for transforming algebraical expressions from Derive were used. Finally, an algebraical equation was obtained describing the torque $M_0$. The result of the solution to this equation was an expression for the dimensionless torque

$$\overline{M} = \frac{M_0}{Eh^3} = \frac{1}{12(1-\nu^2)}C_1 + C_2\Big(\frac{a}{h}\Big)^2 \tag{3.9}$$

where $C_i$ are constants depending on the angle $\theta_1$ and the number $m$. Their form is very complex.

Equation (3.9) was then converted to the form with a dimensionless tangent stress

$$t = \frac{\tau_0}{E} = \frac{\overline{M}}{2\pi\sin^2\theta_1}\Big(\frac{h}{R}\Big)^2 = \frac{\Big(\frac{h}{R}\Big)^2}{2\pi\sin^2\theta_1}\Big[\frac{1}{12(1-\nu^2)}C_1 + C_2\Big(\frac{a}{h}\Big)^2\Big] \tag{3.10}$$

where

$$\tau_0 = \frac{M_0}{2\pi R^2 h\sin^2\theta_1}$$

The dimensionless stress written by equation (3.13) depends on the number $m$ describing the form of the loss of stability. The equation enables one to find the minimal value $t_{min}$ with respect to $m$. The minimal value of the stress is equal to the critical stress $t_{cr}$, while the number $m$ determining the stress is equal to $m_{cr}$.

## 4. Example of calculation

The search for the critical load is practically only possible with a numerical method. For this purpose the $C_i$ constants must be calculated for a pre-determined value of the $\theta_1$ angle and some numbers $m$. The constants $C_i$ were calculated by the procedures included in the Derive. Table 1 presents examples of the constants for $\theta_1 = \pi/12$ and $\theta_1 = \pi/4$.

**Table 1**

| | $\theta_1 = \pi/12$ | | $\theta_1 = \pi/4$ | |
|---|---|---|---|---|
| $M$ | $C_1$ | $C_2$ | $C_1$ | $C_2$ |
| 1 | 64.1419 | 0.4631 | 226.1292 | 0.1896 |
| 2 | 62.2741 | 3.7563 | 154.5125 | 1.9270 |
| 3 | 113.4468 | 12.4487 | 170.8259 | 6.1066 |
| 4 | 221.0470 | 29.2795 | 233.6061 | 14.0478 |

The mathematical program Derive for Windows was applied to draw the diagrams shown in Fig. 2. Figure 2 shows the plots against the co-ordinates $t - a/h$ for $\theta_1 = \pi/12$, $h/R = 0.005$ and $\nu = 0.3$, with various numbers $m$. As shown, the minimal value of $t$ corresponds to $m = 2$.

**Table 2**

| $\theta_1$ | $m_{cr}$ | $t(R/h)^2$ |
|---|---|---|
| $\pi/12$ | 2 | 13.5497 |
| $\pi/10$ | 2 | 10.0269 |
| $\pi/8$ | 2 | 7.4554 |
| $\pi/7$ | 2 | 5.0503 |
| $\pi/5$ | 2 | 4.4628 |
| $\pi/4$ | 2 | 4.5260 |

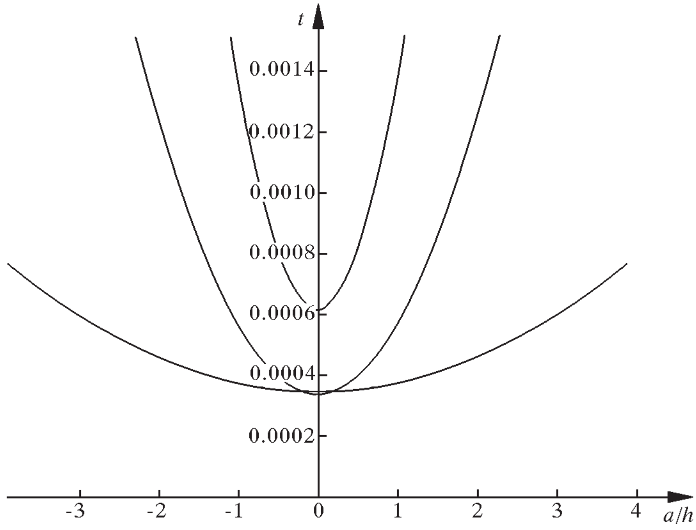


Fig. 2.

Table 2 specifies the dimensionless critical stresses of the shells for some angles $\theta_1$. It is evident that for any $\theta_1$ the number determining the form of the stability loss is equal to $m = m_{cr} = 2$. It should be noticed that for $\theta_1 = \pi/6$, $\theta_1 = 3\pi/10$, and $\theta_1 = 5\pi/14$ the problem remains unsolved. This is certainly a result of the assumed forms of the deflection function and the force-function.

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**Nieliniowe zagadnienie stateczności powłoki kulistej obciążonej momentem obrotowym**

Streszczenie

Cienkościenna powłoka kulista jest podparta przegubowo na obu brzegach. Jeden z brzegów ma możliwość obrotu wokół osi powłoki; do tego brzegu przyłożony jest moment obrotowy. Rozpatruje się zagadnienie stateczności powłoki. Układ równań zagadnienia tworzą nieliniowe równanie równowagi oraz nieliniowe równanie nierozdzielności. Oba równania rozwiązuje się metodą Bubnowa-Galerkina, przyjmując uprzednio postać funkcji ugięcia i funkcji sił. Efektem rozwiązania jest równanie algebraiczne na bezwymiarowy parametr obciążenia. Z tego równania wyznacza się parametr obciążenia krytycznego, odpowiadający minimalnej jego wartości. Liczba $m$, przy której parametr obciążenia osiąga minimum wyznacza postać utraty stateczności. Praca kończy się przykładem liczbowym.

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