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APPROXIMATE CONSTRAINED CONTROLLABILITY OF MECHANICAL SYSTEM[1]

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In the present paper approximate constrained controllability of linear abstract second-order infinite-dimensional dynamical control systems is considered. First, fundamental definitions and notions are recalled. Next it is proved, using the so-called frequency-domain method, that approximate constrained controllability of second-order dynamical control system can be verified by the approximate constrained controllability conditions for the simplified, suitably defined first-order linear dynamical control system. General results are then applied for approximate constrained controllability investigation of mechanical flexible structure vibratory dynamical system. Some special cases are also considered. Moreover, many remarks, comments and corollaries on the relationships between different concepts of approximate controllability are given. Finally, the obtained results are applied for investigation of approximate constrained controllability for flexible mechanical structure. In this case linear second-order partial differential state equation describes the transverse motion of an elastic beam which occupies the given finite interval.

Key words: linear infinite-dimensional control systems, mechanical flexible structure vibratory systems, controllability of abstract dynamical systems

1. Introduction

Controllability, similarly to stability and observability is one of the fundamental concepts in mathematical control theory (Huang, 1988). Roughly speaking, controllability generally means, that it is possible to steer a given dynamical system from an arbitrary initial state to an arbitrary final

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state using the control taken from the set of admissible controls. Therefore, controllability of dynamical system depends on the one side on the form of the strate equation and on the other side-on the set of admissible controls. In the literature, there are many different definitions and conditions of controllability, which depend on the class of dynamical system (Ahmed and Xiang, 1996; Huang, 1988; Klamka, 1992, 1993a,b; Kunimatsu and Ito, 1988; Narukawa, 1982; O'Brien, 1979; Triggiani, 1975b, 1977). Moreover, it should be pointed out, that for infinite-dimensional dynamical systems, it is necessary to distinguish between the notions of approximate controllability and exact controllability (Huang, 1988; Klamka, 1993a, O'Brien, 1979; Triggiani, 1975a,b, 1976, 1977, 1978; Triggiani and Lasiecka, 1991). It follows directly from the fact, that in infinite-dimensional spaces there exist linear subspaces that are not closed. Finally, it should be mentioned, that most of the papers concerning different controllability problems are mainly devoted to a study of unconstrained controllability, i.e. when the values of admissible controls are unconstrained. However, in the papers (Klamka, 1992, 1993a,b) several necassary and sufficient conditions for constrained approximate controllability for linear dynamical systems are formulated and proved.

The present paper is devoted to the study of approximate controllability of linear infinite-dimensional second-order dynamical systems with damping and with constrained set of admissible controls. For such dynamical systems direct verification of approximate constrained controllability is possible but it is rather very difficult and complicated (Klamka, 1991). Therefore, using the frequency-domain method (Klamka, 1993b), it is shown that approximate constrained controllability of second-order dynamical system can be verified by the approximate constrained controllability condition for suitably defined, simplified first-order dynamical system.

The paper is organized as follows. Section 2 contains systems descriptions and fundamental results concerning linear self-adjoint operators. In Section 3 constrained approximate controllability problem for general linear second-order infinite-dimensional control systems with constant coefficients is discussed. The Section 4 is devoted to a detailed study of constrained approximate controllability of certain flexible mechanical control system. In this case, linear second-order partial differential state equation describes the transverse motion of an elastic beam which occupies the given finite interval (Kobayashi, 1992). The solution of the state equation denotes the displacement from the reference state at a given time and at a given space variable. In the state equation, the first term is introduced by accounting rotational forces, next terms with the first-order derivative with respect to time represent internal structural viscous damping, and the last term represents the effect of axial force on the beam (Kobayashi, 1992). Moreover, the boundary conditions correspond to hinged ends of the beam. The special attention is paid to the so-called positive approximate controllability, i.e. approximate controllability with positive controls. Finally, concluding remarks are presented.

2. System description

First of all let us introduce notations and concepts taken directly from the theory of linear opetarors.

Let V and U denote separable Hilbert spaces. Let $A : V \supset D(\mathbf{A}) \to V$ be a linear generally unbounded self-adjoint and positive-definite linear operator with dense domain $D(\mathbf{A})$ in V and compact resolvent $R(s;\mathbf{A}) = (s\mathbf{I} - \mathbf{A})^{-1}$ for all s in the resolvent set $\rho(\mathbf{A})$. Then operator $\mathbf{A}$ has the following properties (Ahmed and Xiang, 1996; Huang, 1988; Kobayashi, 1992; O'Brien, 1979, Triggiani, 1975b):

- Operator $\mathbf{A}$ has only pure discrete point spectrum $\sigma_p(\mathbf{A})$ consisting entirely of isolated real positive eigenvalues s_i such that

  $$0 < s_1 < s_2 < \ldots < s_i < s_{i+1} < \ldots \qquad \lim_{i\to\infty} s_i = +\infty$$

  Each eigenvalue s_i has finite multiplicity $n_i < \infty$ $(i = 1, 2, \ldots)$ equal to the dimensionality of the corresponding eigenmanifold.

- The eigenvectors $\boldsymbol{v}_{ik} \in D(\mathbf{A})$ $(i = 1, 2, \ldots;\ k = 1, 2, \ldots, n_i)$ form a complete orthonormal system in the separable Hilbert space V.

- $\mathbf{A}$ has the spectral representation

  $$\mathbf{A}\boldsymbol{v} = \sum_{i=1}^{\infty} s_i \sum_{k=1}^{n_i} \langle \boldsymbol{v}, \boldsymbol{v}_{ik}\rangle_V \boldsymbol{v}_{ik} \qquad \text{for} \quad \boldsymbol{v} \in D(\mathbf{A})$$

- Fractional powers $\mathbf{A}^{\alpha}$ $(0 < \alpha \leqslant 1)$ of the operator $\mathbf{A}$ can be defined as follows

  $$\mathbf{A}^{\alpha}\boldsymbol{v} = \sum_{i=1}^{\infty} s_i^{\alpha} \sum_{k=1}^{n_i} \langle \boldsymbol{v}, \boldsymbol{v}_{ik}\rangle_V \boldsymbol{v}_{ik} \qquad \text{for} \quad \boldsymbol{v} \in D(\mathbf{A}^{\alpha})$$

where

$$D(\mathbf{A}^{\alpha}) = \Big\{ \boldsymbol{v} \in V: \ \sum_{i=1}^{\infty} (s_i^{\alpha})^2 \sum_{k=1}^{n_i} |\langle \boldsymbol{v}, \boldsymbol{v}_{ik} \rangle_V|^2 < \infty \Big\}$$

- Operators $\mathbf{A}^{\alpha}$ $(0 < \alpha \leqslant 1)$ are self-adjoint, positive-definite with dense domains in V and generate analytic semigroups on V.

Now, let us consider linear infinite-dimensional control system described by the following abstract second-order differential state equation

$$\begin{aligned} &(e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})\ddot{\boldsymbol{v}}(t) + 2(c_2\mathbf{A} + c_1\mathbf{A}^{\frac{1}{2}} + c_0\mathbf{I})\dot{\boldsymbol{v}}(t) + \\ &+(d_2\mathbf{A} + d_1\mathbf{A}^{\frac{1}{2}} + d_0\mathbf{I})\boldsymbol{v}(t) = \mathbf{B}\boldsymbol{u}(t) \end{aligned} \tag{2.1}$$

where $e_2 \geqslant 0$, $e_1 \geqslant 0$, $e_0 \geqslant 0$, $e_2 + e_1 + e_0 > 0$, $c_2 \geqslant 0$, $c_1 \geqslant 0$, $c_0 \geqslant 0$, d_1 and d_0 unrestricted in sign, $d_2 > 0$ are given real constants.

It is assumed that the operator $\mathbf{B}: \ U \to V$ is linear and its adjoint operator $\mathbf{B}^*: V \to U$ is $\mathbf{A}^{\frac{1}{2}}$-bounded (Ahmed and Xiang, 1996; Bensoussan *et al.*, 1993; Klamka, 1993b), i.e. $D(\mathbf{B}^*) \supset D(\mathbf{A}^{\frac{1}{2}})$ and there is a positive real number M such that

$$\|\mathbf{B}^*\boldsymbol{v}\|_U \leqslant M\Big(\|\boldsymbol{v}\|_V + \|\mathbf{A}^{\frac{1}{2}}\boldsymbol{v}\|_V\Big) \qquad \text{for} \quad \boldsymbol{v} \in D(\mathbf{A})$$

Let $\Omega \subset U$ be a convex cone with vertex at the origin in U such that int co $\Omega \neq \emptyset$. In the sequel it is generally assumed, that the admissible controls $\boldsymbol{u} \in L^2_{loc}([0,\infty), \Omega)$. For the set Ω we define the polar cone by $\Omega^o = \{\boldsymbol{w} \in U, \langle \boldsymbol{w}, \boldsymbol{v} \rangle_U \leqslant 0 \text{ for all } \boldsymbol{v} \in \Omega\}$. The closure, the convex hull and the interior are denoted respectively by cl Ω, co Ω and int Ω. The linear subspace spanned by Ω is denoted by span Ω.

It is well known (Bensoussan *et al.*, 1993; Chen and Russell, 1982; Chen and Triggiani, 1989, 1990a,b) that linear abstract ordinary differential equation (2.1) with initial conditions

$$\boldsymbol{v}(0) \in D(\mathbf{A}) \qquad\qquad \dot{\boldsymbol{v}}(0) \in V$$

has for each $t_1 > 0$ and admissible control $\boldsymbol{u} \in L^2_{loc}([0,\infty), \Omega)$ an unique solution $\boldsymbol{v}(t; \boldsymbol{v}(0), \dot{\boldsymbol{v}}(0), \boldsymbol{u}) \in C^2([0,t_1], V)$ such that $\boldsymbol{v}(t) \in D(\mathbf{A})$ and $\dot{\boldsymbol{v}}(t) \in D(\mathbf{A})$ for $t \in (0, t_1]$.

Moreover, for $\boldsymbol{v}(0) \in V$ there exists so-called "mild solution" for the equation (2.1) in the product space $W = V \times V$ with inner product defined as follows

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_W = \langle [\boldsymbol{v}_1, \boldsymbol{v}_2], [\boldsymbol{w}_1, \boldsymbol{w}_2] \rangle_W = \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_V + \langle \boldsymbol{v}_1, \boldsymbol{w}_1 \rangle_V$$

In order to transform the second-order equation (2.1) into the first-order equation in the Hilbert space W, let us make the substitution (Ahmed and Xiang, 1996; Bensoussan *et al.*, 1993; Chen and Russell, 1982; Chen and Triggiani, 1989, 1990a,b; Triggiani, 1977)

$$\boldsymbol{v}(t) = \boldsymbol{w}_1(t) \qquad \dot{\boldsymbol{v}}(t) = \boldsymbol{w}_2(t)$$

Then equation (2.1) becomes

$$\boldsymbol{w}(t) = \mathbf{F}\boldsymbol{w}(t) + \mathbf{G}\boldsymbol{u}(t) \tag{2.2}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{F}_0^{-1}(d_2\mathbf{A} + d_1\mathbf{A}^{\frac{1}{2}} + d_0\mathbf{I}) & -2\mathbf{F}_0^{-1}(c_2\mathbf{A} + c_1\mathbf{A}^{\frac{1}{2}} + c_0\mathbf{I}) \end{bmatrix}$$

$$\boldsymbol{w}(t) = \begin{bmatrix} \boldsymbol{w}_1(t) \\ \boldsymbol{w}_2(t) \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_0^{-1}\mathbf{B} \end{bmatrix}$$

and $\mathbf{F}_0 = (e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})$.

Remark 2.1. Since the operators $\mathbf{A}$ and $\mathbf{A}^{\frac{1}{2}}$ are self-adjoint and under assumptions on coefficients e_i $(i = 0,1,2)$, the sequence $\{(e_2 s_i + e_1\sqrt{s_i} + e_0)^{-1} \in R,\ i = 1,2,\ldots\}$ converges towards zero, it is easy to see that operator $(e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})^{-1}$ is *self-adjoint, positive-definite* and *bounded* on V.

Taking advantage of relation $\langle \boldsymbol{v}_1, \mathbf{F}^*\boldsymbol{v}_2\rangle_W = \langle \mathbf{F}\boldsymbol{v}_1, \boldsymbol{v}_2\rangle_W$, we can obtain for the operator $\mathbf{F}$ its adjoint operator $\mathbf{F}^*$ as follows

$$\mathbf{F}^* = \begin{bmatrix} \mathbf{0} & -(d_2\mathbf{A} + d_1\mathbf{A}^{\frac{1}{2}} + d_0\mathbf{I})(e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})^{-1} \\ \mathbf{I} & -2(c_2\mathbf{A} + c_1\mathbf{A}^{\frac{1}{2}} + c_0\mathbf{I})(e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})^{-1} \end{bmatrix}$$

Similarly, the adjoint for operator $\mathbf{G}$ can be obtained as

$$\mathbf{G}^* = \begin{bmatrix} \mathbf{0} & \mathbf{B}^*(e_2\mathbf{A} + e_1\mathbf{A}^{\frac{1}{2}} + e_0\mathbf{I})^{-1} \end{bmatrix}$$

Remark 2.2. It should be pointed out, that properties of operators $\mathbf{F}$ and $\mathbf{F}^*$ depend strongly on the values of coefficients c_i, d_i, e_i $(i = 0,1,2)$ (Bensoussan *et al.*, 1993; Chen and Russell, 1982; Chen and Triggiani, 1989, 1990a,b). In particular:

1. If $c_2 = c_1 = c_0 = 0$ and additionally
   (a) $e_2 \neq 0$ or ($e_2 = 0$ and $d_2 = 0$ and $e_1 \neq 0$) or ($e_2 = e_1 = 0$ and $d_2 = d_1 = 0$), then the operator $\mathbf{F}$ is bounded and generates an analytic group of linear bounded operators on the Hilbert space $W = V \times V$.
   (b) ($e_2 = 0$ and $d_2 \neq 0$) or ($d_2 = 0$ and $e_2 = e_1 = 0$ and $d_1 \neq 0$), then the operator $\mathbf{F}$ is unbounded and generates a group of linear bounded operators on the Hilbert space $W = V \times V$ which cannot be analytic (Triggiani, 1975b).
2. If ($e_2 = 0$ and $c_2 \neq 0$) or ($e_2 = e_1 = 0$ and ($c_2 \neq 0$ or $c_1 \neq 0$)), then the operator $\mathbf{F}$ is unbounded and generates an analytic semigroup of linear bounded operators on the Hilbert space $W = V \times V$.
3. Moreover, if $e_2 \neq 0$ or ($e_2 = e_1 = 0$ and $c_2 = c_1 = 0$ and $d_2 = d_1 = 0$) or ($e_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_1 \neq 0$), then the operator $\mathbf{F}$ is bounded and generates an analytic semigroup of linear bounded operators on the Hilbert space $W = V \times V$.
4. If $c_2 = e_2 = 0$ and $e_1 \neq 0$ and $d_2 \neq 0$, then the operator $\mathbf{F}$ is unbounded and generates an C_0-semigroup of linear unbounded operators on the Hilbert space $W = V \times V$ which is not analytic.

In the sequel, in addition to the second-order equation (2.1), we shall also consider the simplified first-order linear differential equation of the following form

$$\dot{\boldsymbol{v}}(t) = -\mathbf{A}^{\alpha}\boldsymbol{v}(t) + \mathbf{B}\boldsymbol{u}(t) \tag{2.3}$$

where constant $\alpha \in (0, \infty)$ is such that there exists solution of differential equation (2.3).

In the next sections we shall also consider dynamical control system (2.1) with finite-dimensional control space $U = R^m$. In this special case, for convenience, we shall introduce the following notations

$$\mathbf{B} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_j & \cdots & \boldsymbol{b}_m \end{bmatrix} \qquad \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_j(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

where for $\boldsymbol{b}_j \in V$ for $j = 1, 2, \ldots, m$ and $\boldsymbol{u} \in L^2_{loc}([0, \infty), \Omega)$.

Let us observe, that in this special case linear bounded operator $\mathbf{B}$ is finite-dimensional and therefore, it is a compact operator (Ahmed and Xiang, 1996; Huang, 1988; Triggiani, 1975a, 1976).

Using eigenvectors $\boldsymbol{v}_{ik}$ ($i = 1, 2, \dots$ and $k = 1, 2, \dots, n_i$) we introduce for finite-dimensional operator $\mathbf{B}$ the following notation (Huang, 1988; Triggiani, 1975b) for $i = 1, 2, \dots$

$$\mathbf{B}_i = \begin{bmatrix} \langle \boldsymbol{b}_1, \boldsymbol{v}_{i1} \rangle_V & \langle \boldsymbol{b}_2, \boldsymbol{v}_{i1} \rangle_V & \cdots & \langle \boldsymbol{b}_j, \boldsymbol{v}_{i1} \rangle_V & \cdots & \langle \boldsymbol{b}_m, \boldsymbol{v}_{i1} \rangle_V \\ \langle \boldsymbol{b}_1, \boldsymbol{v}_{i2} \rangle_V & \langle \boldsymbol{b}_2, \boldsymbol{v}_{i2} \rangle_V & \cdots & \langle \boldsymbol{b}_j, \boldsymbol{v}_{i2} \rangle_V & \cdots & \langle \boldsymbol{b}_m, \boldsymbol{v}_{i2} \rangle_V \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{b}_1, \boldsymbol{v}_{ik} \rangle_V & \langle \boldsymbol{b}_2, \boldsymbol{v}_{ik} \rangle_V & \cdots & \langle \boldsymbol{b}_j, \boldsymbol{v}_{ik} \rangle_V & \cdots & \langle \boldsymbol{b}_m, \boldsymbol{v}_{ik} \rangle_V \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{b}_1, \boldsymbol{v}_{in_i} \rangle_V & \langle \boldsymbol{b}_2, \boldsymbol{v}_{in_i} \rangle_V & \cdots & \langle \boldsymbol{b}_j, \boldsymbol{v}_{in_i} \rangle_V & \cdots & \langle \boldsymbol{b}_m, \boldsymbol{v}_{in_i} \rangle_V \end{bmatrix} \tag{2.4}$$

$\mathbf{B}_i$ ($i = 1, 2, \dots$) are $n_i \times m$-dimensional constant matrices which play an important role in approximate controllability investigations (Huang, 1988; Klamka, 1991, 1993a; Triggiani, 1975b).

For the special case when eigenvalues s_i are simple, i.e. $n_i = 1$ ($i = 1, 2, \dots$) and consequently the matrices $\mathbf{B}_i$ are in fact m-dimensional row vectors for $i = 1, 2, \dots$

$$\mathbf{B}_i = \Big[\langle \boldsymbol{b}_1, \boldsymbol{v}_i \rangle_V \quad \cdots \quad \langle \boldsymbol{b}_j, \boldsymbol{v}_i \rangle_V \quad \cdots \quad \langle \boldsymbol{b}_m, \boldsymbol{v}_i \rangle_V \Big] \tag{2.5}$$

3. Constrained approximate controllability

It is well known, that for infinite-dimensional dynamical systems we may introduce two general kinds of controllability, i.e. approximate (weak) controllability and exact (strong) controllability (Ahmed and Xiang, 1996; Huang, 1988; Klamka, 1993a; Triggiani, 1975a, 1976). However, it should be mentioned, that in the case when the linear semigroup associated with the dynamical system is a compact semigroup or the control operator $\mathbf{B}$ is compact, then dynamical system is never exactly controllable in infinite-dimensional state space (Ahmed and Xiang, 1996; Huang, 1988; Triggiani, 1975b, 1977). Therefore, in the present paper we shall concentrate on approximate controllability for second-order dynamical system (2.1) or equivalently (2.2), and first of all we recall the basis definition. Next, we shall recall from the literature several lemmas and controllability conditions which will be used to verify the constrained approximate controllability of certain mechanical system.

Definition 3.1 (Ahmed and Xiang, 1996; Huang, 1988; O'Brien, 1979). Dynamical system (2.1) is said to be *Ω-approximately controllable* if for any initial condition $\boldsymbol{w}(0) \in V \times V$, any given final condition $\boldsymbol{w}_f \in V \times V$ and each positive real number ε, there exists a finite time $t_1 < \infty$ (depending generally on $\boldsymbol{w}(0)$ and $\boldsymbol{w}_f$) and an admissible control $\boldsymbol{u} \in L^2([0,t_1],\Omega)$ such that

$$\|\boldsymbol{w}(t_1;\boldsymbol{w}(0),\boldsymbol{u}) - \boldsymbol{w}_f\|_{V\times V} \leqslant \varepsilon$$

Now, let us recall several well-known lemmas (Klamka, 1993a,b; Narukawa, 1984; O'Brien, 1979) concerning constrained approximate controllability of the first-order linear infinite-dimensional dynamical system (2.2), which will be useful in the sequel.

Lemma 3.1. (Klamka, 1993b). Dynamical system (2.2) is U-approximately controllable if and only if for any complex number z, there exists no nonzero $\boldsymbol{w} \in D(\mathbf{F}^*)$ such that

$$\begin{bmatrix} \mathbf{F}^* - z\mathbf{I} \\ \mathbf{G}^* \end{bmatrix} \boldsymbol{w} = \mathbf{0} \tag{3.1}$$

Similarly, dynamical system (2.3) is U-approximately controllable if and only if for any complex number s there exists no nonzero $\boldsymbol{v} \in D(\mathbf{A}^\alpha) \subset V$ such that

$$\begin{bmatrix} \mathbf{A}^\alpha - s\mathbf{I} \\ \mathbf{B}^* \end{bmatrix} \boldsymbol{v} = \mathbf{0}$$

Lemma 3.2. (Klamka, 1993a). Suppose that $U = R^m$, and the cone $\Omega = \{\boldsymbol{u} \in R^m = U : \ u_j(t) \geqslant 0, \ \text{for } t \geqslant 0\}$, then dynamical system (2.3) is Ω-approximately controllable if and only if the columns of the matrices $\mathbf{B}_i$ form a positive basis in the space R^{n_i} for every $i = 1,2,\ldots$.

Lemma 3.3. (Narukawa, 1984). Dynamical system (2.3) is U-approximately controllable if and only if it is approximately controllable for some $\alpha \in (0,\infty)$.

Lemma 3.4. (O'Brien, 1979). Dynamical system (2.2) is Ω-approximately controllable if and only if it is U-approximately controllable and

$$\operatorname{Ker}(z\mathbf{I} - \mathbf{F}^*) \cap (\mathbf{G}\Omega)^o = \{0\} \qquad \text{for every} \quad z \in R \tag{3.2}$$

Remark 3.1. Since the linear operator $\mathbf{A}$ is selfadjoint then from Lemmas 3.1, 3.3, and 3.4 directly follows that the dynamical system (2.3) is Ω-approximately controllable if and only if

$$\mathrm{Ker}(s\mathbf{I} - \mathbf{A}^{\alpha}) \cap (\mathbf{B}\Omega)^{o} = \{0\} \qquad \text{for every} \quad s \in R \tag{3.3}$$

Proposition 3.1. Dynamical system (2.3) is Ω-approximately controllable if and only if it is Ω-approximately controllable for some $\alpha \in (0,\infty)$.

Proof. Since the operator $\mathbf{A}$ is selfadjoint and positive definite, then for any real number $\alpha \in (0,\infty)$

$$\mathrm{Ker}(s\mathbf{I} - \mathbf{A}) = \mathrm{Ker}(s^{\alpha}\mathbf{I} - \mathbf{A}^{\alpha}) = \mathrm{Ker}(z\mathbf{I} - \mathbf{A}^{\alpha})$$

where $z = s^{\alpha}$ is a homeomorphizm. Hence our proposition follows.

Now, using the frequency-domain method (Klamka, 1993b) we shall formulate the necessary and sufficient condition for approximate controllability of dynamical system (2.1), which is proved by Klamka (1993a).

Theorem 3.1. (Klamka, 1993a). Dynamical system (2.1) is Ω-approximately controllable if and only if dynamical system (2.3) is Ω-approximately controllable for some $\alpha \in (0,\infty)$.

From Theorem 3.1 follow several Corollaries, which are necessary and sufficient conditions for constrained approximate controllability for different special cases of dynamical system (2.1).

Corollary 3.1. Suppose that $\Omega = \{\boldsymbol{u} \in R^m = U : \; u_j(t) \geqslant 0, \text{ for } t \geqslant 0\}$. Then the dynamical control system (2.1) is Ω-approximately controllable, i.e. with positive controls if and only if columns of the matrices $\mathbf{B}_i$ form a positive basis in the space R^{n_i} for every $i = 1, 2, \ldots$.

Proof. If the columns of the matrices $\mathbf{B}_i$ form a positive basis in the space R^n for every $i = 1, 2, \ldots$ and Ω is a positive cone in the space R^m, then image $\mathbf{B}\Omega$ is the whole space R^{n_i} for every $i = 1, 2, \ldots$. Therefore our Corollary 3.1 follows.

Corollary 3.2. Suppose that $c_1^2 + c_2^2 > 0$ and $\Omega = U$. Then dynamical system (2.1) is U-approximately controllable, i.e. without control constraints in any time interval $[0, t_1]$ if and only if dynamical system (2.3) is U-approximately controllable in finite time.

Proof. Since for the case when $c_1^2 + c_2^2 > 0$ operator F generates analytic semigroup, then approximate controllability of dynamical system (2.2) and hence also of dynamical system (2.1) is equivalent to its approximate controllability in any time interval $[0, t_1]$ (Klamka, 1993a; Triggiani, 1977). Therefore, from Theorem 3.1 immediately follows Corollary 3.2.

Corollary 3.3. Suppose that $c_1^2 + c_2^2 > 0$, $\Omega = U$, and the space of control values is finite-dimensional, i.e. $U = R^m$. Then the dynamical system (2.1) is U-approximately controllable, i.e. without control constraints in any time interval $[0, t_1]$ if and only if

$$\operatorname{rank} \mathsf{B}_i = n_i \qquad \text{for} \quad i = 1, 2, \ldots$$

Proof. Corollary 3.3 is a direct consequence of the Theorem 3.1, Corollary 3.2 and well-known results (Huang, 1988; Triggiani, 1975a,b, 1976) concerning approximate controllability of infinite-dimensional dynamical systems with finite-dimensional controls.

Corollary 3.4. Suppose that $c_1^2 + c_2^2 > 0$, $\Omega = U$, the space of control values is finite-dimensional, i.e. $U = R^m$, and moreover, multiplicities $n_i = 1$ for $i = 1, 2, \ldots$. Then dynamical control system (2.1) is U-approximately controllable, i.e. without control constraints in any time interval $[0, t_1]$ if and only if

$$\sum_{j=1}^{m} \langle \boldsymbol{b}_j, \boldsymbol{v}_i \rangle_V^2 \neq 0 \qquad \text{for} \quad i = 1, 2, \ldots$$

Proof. From Corollary 3.3 immediately follows that for the case when multiplicities $n_i = 1$ for $i = 1, 2, \ldots$ dynamical system (2.1) is U-approximately controllable in any time interval if and only if m-dimensional row vectors for $i = 1, 2, \ldots$

$$\mathsf{B}_i = \Big[\langle \boldsymbol{b}_1, \boldsymbol{v}_i \rangle_V \quad \langle \boldsymbol{b}_2, \boldsymbol{v}_i \rangle_V \quad \ldots \quad \langle \boldsymbol{b}_j, \boldsymbol{v}_i \rangle_V \quad \ldots \quad \langle \boldsymbol{b}_m, \boldsymbol{v}_i \rangle_V \Big]$$

Thus, Corollary 3.4 follows.

In the next section we shall use the general controllability results given above to verify approximate constrained controllability of a certain vibratory dynamical system modeling mechanical flexible structure.

4. Approximate constrained controllability of vibratory system

In this section we shall consider a vibratory dynamical system described by the following linear partial differential state equation (Kobayashi, 1992)

$$\begin{aligned}&e_1v_{ttxx}(t,x)+e_0v_{tt}(t,x)+2c_1v_{txx}(t,x)+2c_2v_{txxxx}(t,x)+\\&+d_1v_{xx}(t,x)+d_2v_{xxxx}(t,x)=\sum_{j=1}^{m}b_j(x)u_j(t)\end{aligned} \tag{4.1}$$

defined for $x\in[0,L]$ and $t\in[0,\infty)$, where the subscript t represents partial derivative with respect to time variable, while x denotes partial derivative with respect to spatial coordinate.

The initial conditions for the equation (4.1) are given by

$$v(0,x)=v_0(x)\quad\text{and}\quad v_t(0,x)=v_1(x)\quad\text{for}\quad x\in[0,L] \tag{4.2}$$

and boundary conditions are as follows

$$v(t,0)=v(t,L)=v_{xx}(t,0)=v_{xx}(t,L)=0\qquad\text{for}\quad t\in[0,\infty) \tag{4.3}$$

Let Ω be the positive cone $\Omega=\{\boldsymbol{u}\in R^m=U:\ u_j(t)\geqslant 0,\ \text{for } t\geqslant 0\}$, i.e. in the sequel we shall consider mechanical system with positive controls.

It should be stressed, that the partial differential state equation (4.1) describes the transverse motion of an elastic beam which occupies the interval $[0,L]$ in the reference and stress-free state. The function $v(t,x)$ denotes the displacement from the reference state at time t and position x. In the left-hand side of the equation (4.1), the first term is introduced by accounting rotational forces, terms with the first-order derivative with respect to time represent internal structural viscous damping, and the fifth term represents the effect of axial force on the beam (Kobayashi, 1992). The boundary conditions (4.3) correspond to hinged ends of the beam.

Let $V=L^2[0,L]$ be a separable Hilbert space of all square integrable functions on $[0,L]$ with the standard norm and inner product (Ahmed and Xiang, 1996; Huang, 1988). In order to regard the vibratory system (4.1), (4.2) and (4.3) in the general framework considered in the previous sections, let us define linear unbounded differential operator $\mathbf{A}:V\supset D(\mathbf{A})\to V$ by Kobayashi (1992)

$$\begin{aligned}&v(x)=v_{xxxx}(x)\qquad\text{for}\quad v(x)\in D(\mathbf{A})\\&D(\mathbf{A})=\Big\{v(x)\in H^4[0,L];\ v(0)=v(L)=v_{xx}(0)=v_{xx}(L)=0\Big\}\end{aligned} \tag{4.4}$$

where $H^4[0, L]$ denotes the fourth-order Sobolev space on $[0, L]$.

The linear unbounded operator $\mathbf{A}$ has the following properties (Ahmed and Xiang, 1996; Huang, 1988; Kobayashi, 1992; O'Brien, 1979):

- Operator $\mathbf{A}$ is self-adjoint and positive-definite with dense domain $D(\mathbf{A})$ in the space V.

- There exists a compact inverse $\mathbf{A}^{-1}$, and consequently the resolvent $R(s; \mathbf{A})$ of $\mathbf{A}$ is a compact operator for all $s \in \rho(\mathbf{A})$.

- Operator $\mathbf{A}$ has a spectral representation

$$\mathbf{A}v = \sum_{i=1}^{\infty} s_i \langle v, v_i \rangle_V v_i \qquad \text{for} \quad v \in D(\mathbf{A})$$

where $s_i > 0$ $(i = 1, 2, \ldots)$ are simple eigenvalues (i.e. $n_i = 1$) and $v_i \in D(\mathbf{A})$ $(i = 1, 2, \ldots)$ are the corresponding eigenfunctions of $\mathbf{A}$. Moreover, for $x \in [0, L]$

$$s_i = \left(\frac{\pi i}{L}\right)^4 \qquad\qquad v_i(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi i x}{L}$$

and the set $\{v_i(x),\ i = 1, 2, \ldots\}$ forms a complete orthonormal system in V.

- Fractional powers $\mathbf{A}^{\alpha}$, $0 < \alpha \leqslant 1$ can be defined by

$$\mathbf{A}^{\alpha}v = \sum_{i=1}^{\infty} s_i^{\alpha} \langle v, v_i \rangle_V v_i \qquad \text{for} \quad v \in D(\mathbf{A}) \quad (0 < \alpha \leqslant 1)$$

which is also a self-adjoint and positive-definite operator with a dense domain in V. In particular, for we have

$$\mathbf{A}^{\frac{1}{2}} v = -v_{xx}$$

with the domain $D(\mathbf{A}^{\frac{1}{2}}) = \{v \in H^2[0, L] :\ v(0) = v(L)\}$.

Now, we can consider the partial differential equation (4.1) with conditions (4.2) and boundary conditions (4.3) as a special case of the second-order abstract evolution equation (2.1) in the Hilbert space V.

$$(e_1\mathbf{A}^{\frac{1}{2}} + e_0)\ddot{w}(t) + 2(c_2\mathbf{A} + c_1\mathbf{A}^{\frac{1}{2}})\dot{w}(t) + (d_2\mathbf{A} + d_1\mathbf{A}^{\frac{1}{2}})w(t) = \mathbf{B}\boldsymbol{u}(t) \tag{4.5}$$

where

$$w(t) = v(t,\cdot) \in V \qquad \dot{w}(t) = v_t(t,\cdot) \in V \qquad \ddot{w}(t) = v_{tt}(t,\cdot) \in V$$
$$b_j = b_j(\cdot) \in V \qquad (j = 1, 2, \dots, m)$$

Let the initial conditions be of the following form

$$w(0) = w_0 \in D(\mathbf{A}) \qquad \dot{w}(0) = w_1 \in V$$

Then there exists a unique solution of the partial differential equation (4.1) (Kobayashi, 1992).

Now, using the results given in Section 3 we shall formulate and prove the necessary and sufficient condition for approximate controllability of the vibratory dynamical control system (4.1), which is the main result of the present paper.

Theorem 4.1. Vibratory dynamical control system (4.1) is Ω-approximately controllable, i.e. with positive controls if and only if for each $i = 1, 2, \dots$ m-dimensional row vectors $\boldsymbol{B}_i = [b_{i1}, b_{i2}, \dots, b_{ij}, \dots, b_{im}]$ contain at least two coefficients with different signs, where

$$b_{ij} = \int\limits_0^L \sqrt{\frac{2}{L}}\, b_j(x) \sin\frac{\pi i x}{L}\, dx \qquad \begin{array}{l} i = 1, 2, \dots \\ j = 1, 2, \dots, m \end{array} \tag{4.6}$$

Proof. Let us observe, that dynamical system (4.1) satisfies all the assumptions of Corollary 3.1. Therefore, taking into account the analytic formula for the eigenvectors $\boldsymbol{v}_i(x)$, $i = 1, 2, \dots$ and the form of the inner product in the separable Hilbert space $L^2([0, L], R)$, from relation (4.1) we directly obtain inequalities (4.6). Hence, Theorem 4.1 immediately follows.

Corollary 4.1. Vibratory dynamical control system (4.1) is U-approximately controllable, i.e. without control constraints in any time interval $[0, t_1]$ if and only if

$$\sum_{j=1}^m \Big(\int\limits_0^L \sqrt{\frac{2}{L}}\, b_j(x) \sin\frac{\pi i x}{L}\, dx\Big)^2 \neq 0 \qquad \text{for} \quad i = 1, 2, \dots \tag{4.7}$$

Proof. Let us observe, that dynamical system (4.1) satisfies all the assumptions of Corollary 3.4. Therefore, taking into account the analytic formula for the eigenfunctions $v_i(x)$ $(i = 1, 2, \ldots)$ and the form of the inner product in the separable Hilbert space $L^2([0, L], R)$, from Lemma 3.4 we directly obtain inequalities (4.6). Hence, Theorem 4.1 follows immediately.

5. Conclusions

The present paper contains results concerning approximate controllability of second-order abstract infinite-dimensional dynamical systems. Using the frequency-domain method (Klamka, 1993b) and the methods of functional analysis, especially the theory of linear unbounded operators, necessary and sufficient conditions for approximate controllability in any time interval are formulated and proved. Moreover, some special cases are also investigated and discussed. Then, the general controllability conditions are applied to investigate approximate controllability of vibratory dynamical system modeling flexible mechanical structure.

The results presented in the paper are generalization of the controllability conditions given in the literature (Ahmed and Xiang, 1996; Klamka, 1991, 1993b; Narukawa, 1984; O'Brien, 1979; Triggiani, 1975a,b) to second-order abstract dynamical systems with damping terms. Finally, it should be pointed out,that the obtained results could be extended to cover the case of more complicated second-order abstract dynamical systems (Chen and Russel, 1982; Chen and Triggiani, 1989, 1990a,b).

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Przybliżona ograniczona sterowalność układu mechanicznego

Streszczenie

W artykule rozpatrywana jest przybliżona ograniczona sterowalność liniowego abstrakcyjnego nieskończenie-wymiarowego układu dynamicznego drugiego rzędu. W pierwszej kolejności przedstawiono podstawowe definicje i pojęcia. Następnie, wykorzystując metodę częstotliwościową, wykazano, że przybliżona ograniczona sterowalność układu dynamicznego drugiego rzędu może być weryfikowana poprzez badanie przybliżonej ograniczonej sterowalności odpowiednio zdefiniowanego uproszczonego układu dynamicznego pierwszego rzędu. Ogólne metody zastosowano do badania przybliżonej ograniczonej sterowalności mechanicznego układu oscylacyjnego o elastycznej strukturze. Rozpatrzono również pewne przypadki szczególne. Ponadto podano wiele uwag, komentarzy i wniosków dotyczących relacji między różnymi rodzajami przybliżonej sterowalności. Jako przykład zastosowań sformułowano warunki przybliżonej ograniczonej sterowalności w odniesieniu do elastycznego układu mechanicznego. W tym wypadku liniowe równanie różniczkowe cząstkowe stanu opisuje odchylenie elastycznej belki o danej długości.

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