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DAMAGE MODELLING FRAMEWORK FOR VISCOELASTIC PARTICULATE COMPOSITES VIA A SCALE TRANSITION APPROACH

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The aim of this paper is to pursue, in the wake of the work by Nadot-Martin *et al.* (2003), a non-classical micromechanical study and scale transition for highly filled particulate composites with viscoelastic matrices. The present extension of a morphology-based approach due to Christoffersen (1983), carried forward to the viscoelastic small strain context by Nadot-Martin *et al.* (2003), consists here in introducing a supplementary mechanism, namely damage by grain/matrix debonding. Displacement discontinuities (microcracks) on grain/matrix interfaces are first incorporated in a compatible way within geometric and kinematic hypotheses regarding the grains-and-layers assembly of Christoffersen. Then, local field expressions as well as homogenized stresses are established and discussed for a given state of damage (i.e. for a given actual number of open and closed microcracks) and using the hypothesis of no sliding on closed crack lips. A comparison with the results obtained for the sound viscoelastic composite by Nadot-Martin *et al.* (2003) allows to quantify the damage influence on local and global levels. At last, the basic formulation of the model obtained by scale transition is completed by the second stage leading to a thermodynamically consistent formulation eliminating some superfluous damaged-induced strain-like variables related to open cracks. This second stage is presented here for a simplified system where delayed (viscoelastic) effects are (tentatively) neglected. It appears as a preliminary and crucial step for further generalization in viscoelasticity.

Key words: micro-macro transition, heterogeneous materials, morphology, viscoelasticity, anisotropic damage, microcracking

1. Introduction

This paper deals with a two step scale transition for modelling anisotropic damage behaviour of viscoelastic particulate composites, starting from the methodology initially proposed by Christoffersen (1983) for elastic bonded granulates. This methodology is built on geometric and kinematic hypotheses regarding a granular assembly with interconnecting layers constituting thus a consistent framework of a microstructural morphology pattern. The latter forms in fact an advantageous starting point for a micromechanical description and further localization-homogenization procedure. The recent extension of the method, performed by Nadot-Martin *et al.* (2003) for composites involving viscoelastic matrices, has confirmed its efficiency since it allows one to account for genuine viscoelastic interactions between constituents and for their macroscopic consequence – the "long range memory" effect. It is to be recalled that the presence of truly viscoelastic (i.e. viscous and elastic) coupled interactions on the microscale level and of associated global "long range memory" constitute two crucial criteria for relative evaluation of the pertinency of scale transition in the viscoelastic context (see e.g. Beurthey and Zaoui, 2000; Brenner *et al.*, 2002). The present contribution attempts to further extend the technique in the presence of damage by grain-matrix debonding. It is to be emphasized that the resulting two step scale transition presented is done for a given diffuse distribution of open and closed interface microdefects (i.e. without coalescence).

The aim of Section 2 is to extend the technique due to Christoffersen (1983) – with its geometrical and kinematical ingredients, its averaging scheme and the relevant strategy of the approach of the local problem – in the presence of interfacial discontinuities. In such a way, Section 2 provides generalization, involving the damage mechanism mentioned, of the conceptual structure and relevant consistency requirements by Christoffersen. Section 3 deals with the solution to the localization-homogenization problem for composites with a viscoelastic matrix as it was done for the sound aggregate by Nadot-Martin *et al.* (2003), while here it is performed in the presence of interfacial damage. A discussion is put forward (Subsection 3.3) in order to quantify the coupling between damage and viscoelasticity regarding several aspects as e.g. local interactions and the macroscopic consecutive long range memory effect, induced anisotropy, moduli recovery under crack closure. At this stage, local fields and global stresses involve a full set of internal relaxation variables (as for the sound material) and a new set of strain-like variables related to (discrete) sites of microcracking. In the same time, the reversible global moduli tensor lacks crucial symmetries. The discussion at the end of Subsection 3.3 brings out the necessity of complementary analysis in order to express local open defects-related strains as functions of macroscopic state variables. Nevertheless, the simultaneous presence of viscoelastic variables and damage related ones makes the problem complex to deal with. This is why the above mentioned specific analysis, called the 'complementary localization-homogenization approach' is conducted here (Section 4) for an elastic aggregate only (elastic grains and matrix + microcracks open/closed). This is a (necessary) crucial step, and the results obtained will constitute the basis for further genuine viscoelastic analysis.

2. Extension of Christoffersen's method in presence of damage

2.1. Microstructure schematization

Figure 1 shows a close-up schematic for grains separated by matrix layers according to the scheme proposed by Christoffersen (1983) for a sound, i.e. an undamaged particulate composite. The grains are considered as polyhedral; any two of them are interconnected by a thin material layer of a given uniform thickness (noted h^α for the αth layer). The grain-layer interfaces are characterized by their orientation ($\boldsymbol{n}^\alpha$ for the αth layer). The spatial distribution of grains is accounted for through vectors linking grain centroids ($\boldsymbol{d}^\alpha$ for the αth layer). Moreover, no restriction is imposed on the grain size – the representation allows granulometric variations. As a result, such a schematization, giving much attention to the granular character, makes it possible to describe with sufficient accuracy the real initial microstructure geometry of an ample class of strongly charged particulate composites. Moreover, such a direct morphological description will allow one to introduce local interfacial defects (discontinuities) in a relatively direct and simple manner (see Subsection 2.2).

2.2. Local problem approach

2.2.1. Kinematics

The purpose consists here in introducing material discontinuities and relative displacement jumps in a compatible way with the original kinematical framework of Christoffersen (1983), and following step by step the strategy of this author. This implies more detailing of this framework. The latter is

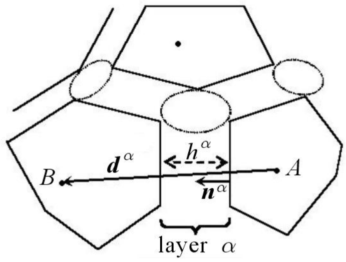

Fig. 1. Two neighbouring grains GA and GB with an interconnecting material layer according to Christoffersen (1983)

defined by four assumptions for the local displacement field that are recalled below:

- The kinematics of grain centroid is characterized by the global (macroscopic) displacement gradient $\nabla \boldsymbol{U} = \mathbf{F}$.
- The grains are supposed homogeneously deformed and the corresponding displacement gradient $\mathbf{f}^0$ is assumed to be common to all members of the Representative Volume Element (RVE).
- Each interconnecting layer is subject to a homogeneous deformation, proper for the layer α under consideration. The corresponding displacement gradient is denoted $\mathbf{f}^\alpha$ for the αth layer.
- Local disturbances at grain edges and corners are neglected on the basis of thinness of the layers (see surrounded zones in Fig. 1).

Following the methodology by Christoffersen, the first stage consists in interpreting the above hypotheses. The resulting three following equations correspond to relations (2.1)-(2.3) in the original paper by Christoffersen (1983). According to the first item, the centroid displacements u_i^A and u_i^B of two grains GA and GB separated by the layer α (see Fig. 1) are given by

$$u_i^A = u_i^0 + F_{ij} y_j^A \qquad\qquad u_i^B = u_i^0 + F_{ij} y_j^B \tag{2.1}$$

where $\boldsymbol{u}^0$ designates a global constant vector and y_j $(j = 1, 2, 3)$ represent local, cartesian coordinates in the RVE. Therefore and with the second assumption, the displacements of the grains GA and GB are

$$\begin{aligned} u_i^{GA}(\boldsymbol{y}) &= u_i^0 + (F_{ij} - f_{ij}^0) y_j^A + f_{ij}^0 y_j \\ u_i^{GB}(\boldsymbol{y}) &= u_i^0 + (F_{ij} - f_{ij}^0) y_j^B + f_{ij}^0 y_j \end{aligned} \tag{2.2}$$

At last, by means of the third assumption, the displacement field of the layer α is

$$u_i^\alpha(\boldsymbol{y}) = u_i^\alpha(\boldsymbol{y}^{AB}) + f_{ij}^\alpha(y_j - y_j^{AB}) \tag{2.3}$$

where AB stands for an arbitrary point on I_1^α, the interface of the layer α and the grain GA.

For a sound material, further developments by Christoffersen consist in expressing $\boldsymbol{u}^\alpha$ and finally the displacement gradient $\mathbf{f}^\alpha$ of any layer α as functions of $\mathbf{F}$, the macroscopic displacement gradient, $\mathbf{f}^0$, the grain displacement gradient and of morphological parameters of the layer

$$f_{ij}^\alpha = f_{ij}^0 + (F_{ik} - f_{ik}^0)d_k^\alpha \frac{n_j^\alpha}{h^\alpha} \tag{2.4}$$

To this aim, Christoffersen employs the continuity of the displacement field successively on I_1^α and I_2^α, namely, makes use of what happens at the grain/layer interfaces. These developments are here revisited to take into account the presence of discontinuities. Following the spirit of the author, it leads to consideration of the jumps (discontinuities) as data of the local problem and to search for $\boldsymbol{u}^\alpha$ and $\mathbf{f}^\alpha$ as functions of these. It is stressed that this is also the option taken by some works regarding homogenization of microcracked solids (Andrieux *et al.*, 1986; Kachanov, 1994; Basista and Gross, 1997) where the local problem is solved by considering the displacement jumps (corresponding to cracks) as the relevant data.

So, consider the presence of a discontinuity on the first interface I_1^α of the layer α. According to the previous remark, the corresponding displacement discontinuity vector, denoted $b_i^{\alpha 1}$, is considered as a data of the local problem. Nevertheless, its form cannot be arbitrary. Indeed, the linearity (according to kinematical assumptions) of the displacement field leads to assignment of a linear form to $b_i^{\alpha 1}$, namely

$$b_i^{\alpha 1}(\boldsymbol{y}^{AB}) = f_{ij}^{\alpha D1} y_j^{AB} + c_i^{\alpha D1} \tag{2.5}$$

where the tensor $\mathbf{f}^{\alpha D1}$ and the vector $\boldsymbol{c}^{\alpha D1}$ are homogeneous and stand for data characterizing the crack. So, in the presence of a discontinuity on the first interface I_1^α, instead of researching u_i^α by means of the continuity condition $u_i^\alpha(\boldsymbol{y}^{AB}) = u_i^{GA}(\boldsymbol{y}^{AB})$ for any point AB on I_1^α as it was done by Christoffersen for the sound material, one has to find it in such a way that

$$u_i^\alpha(\boldsymbol{y}^{AB}) = u_i^{GA}(\boldsymbol{y}^{AB}) + b_i^{\alpha 1}(\boldsymbol{y}^{AB}) \qquad \forall \boldsymbol{y}^{AB} \in I_1^\alpha \tag{2.6}$$

with $b_i^{\alpha 1}$ given by (2.5). By reporting (2.6) where u_i^{GA} and $b_i^{\alpha 1}$ are expressed using $(2.2)_1$ and (2.5) respectively, in (2.3), the displacement field in the layer α is obtained in the following form

$$u_i^\alpha(\boldsymbol{y}) = u_i^0 + (F_{ij} - f_{ij}^0)y_j^A + (f_{ij}^0 - f_{ij}^\alpha)y_j^{AB} + f_{ij}^\alpha y_j + f_{ij}^{\alpha D1} y_j^{AB} + c_i^{\alpha D1} \tag{2.7}$$

As the above expression must be independent of the choice of the point AB, the condition

$$(f_{ij}^0 - f_{ij}^\alpha + f_{ij}^{\alpha D1})m_j^\alpha = 0 \tag{2.8}$$

must hold for any tangent $\boldsymbol{m}^\alpha$ to the grain-layer interface I_1^α. It follows that $\mathbf{f}^\alpha$ must have the form

$$f_{ij}^\alpha = f_{ij}^0 + f_{ij}^{\alpha D1} + g_i^\alpha n_j^\alpha \tag{2.9}$$

where $\boldsymbol{g}^\alpha$ denotes a homogeneous vector. Furthermore, expression (2.7) for u_i^α becomes independent of the choice of the point AB, and is finally given by

$$u_i^\alpha(\boldsymbol{y}) = u_i^0 + (F_{ij} - f_{ij}^0)y_j^A + f_{ij}^0 y_j + g_i^\alpha z^{AB} + f_{ij}^{\alpha D1} y_j + c_i^{\alpha D1} \tag{2.10}$$

where $z^{AB} = n_i^\alpha(y_i - y_i^{AB})$ is the distance from the debonded interface I_1^α. Except for the starting point consisting in satisfying (2.6) – in the place of the displacement continuity for any point AB on I_1^α – the foregoing reasoning in the presence of damage constitutes a simple extension of that advanced by Christoffersen (1983) for the sound composite (see for comparison relations (2.3)-(2.6) in the reference quoted).

For the sound material, further developments by Christoffersen concern the determination of the homogeneous vector $\boldsymbol{g}^\alpha$ regarding the continuity on the second interface I_2^α. In the presence of a discontinuity on the first interface I_1^α, the (basic) hypothesis stipulating a homogenous displacement gradient for two grains separated by the layer α – making two opposite faces deform in the manner to stay parallel – makes necessary to introduce simultaneously a discontinuity on the second interface I_2^α, see Fig. 2.

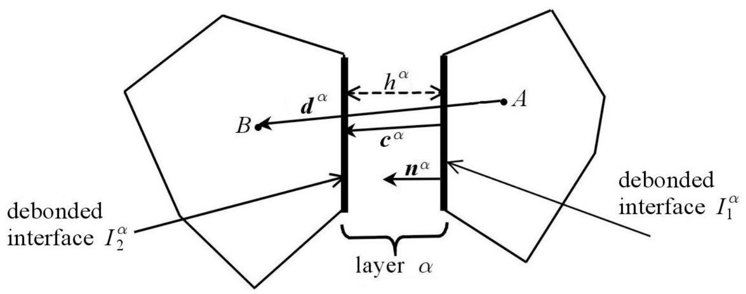

Fig. 2. A layer with cracks at its boundaries

For the same reasons as for $b_i^{\alpha 1}$ on the first interface, one assigns a linear form to the displacement discontinuity vector, noted $b_i^{\alpha 2}$, across I_2^α

$$b_i^{\alpha 2}(\boldsymbol{y}^{BA}) = f_{ij}^{\alpha D2} y_j^{BA} + c_i^{\alpha D2} \tag{2.11}$$

where the tensor $\mathbf{f}^{\alpha D2}$ and the vector $\boldsymbol{c}^{\alpha D2}$ are homogeneous and stand for new data of the local problem that could be *a priori* considered as different from $\mathbf{f}^{\alpha D1}$ and $\boldsymbol{c}^{\alpha D1}$. The vector $\boldsymbol{g}^\alpha$ is then searched in such a way that

$$u_i^\alpha(\boldsymbol{y}^{BA}) = u_i^{GB}(\boldsymbol{y}^{BA}) + b_i^{\alpha 2}(\boldsymbol{y}^{BA}) \qquad \forall \boldsymbol{y}^{BA} \in I_2^\alpha \tag{2.12}$$

be satisfied with $b_i^{\alpha 2}$ given by (2.11) and u_i^α, u_i^{GB} expressed via (2.10) and $(2.2)_2$, respectively. In this manner, one mathematically proves that $\mathbf{f}^{\alpha D1} = \mathbf{f}^{\alpha D2}$. It is stressed that this relation is not a choice but a consequence of the methodology assumed. In the following, this displacement gradient will be denoted $\mathbf{f}^{\alpha D}$. The finally obtained form of $\boldsymbol{g}^\alpha$ allows one to express $\mathbf{f}^\alpha$ in the following manner

$$f_{ij}^\alpha = f_{ij}^0 + (F_{ik} - f_{ik}^0) d_k^\alpha \frac{n_j^\alpha}{h^\alpha} + f_{ij}^{\alpha D} + (c_i^{\alpha D2} - c_i^{\alpha D1}) \frac{n_j^\alpha}{h^\alpha} \tag{2.13}$$

The supplementary terms in (2.13) compared to (2.4) represent a specific contribution of two microcracks located at the boundaries of the debonded layer α considered.

At this stage, all the ingredients of the methodology by Christoffersen in order to express $\mathbf{f}^\alpha$ have been exploited, and it appears necessary to recapitulate different implications of the kinematical hypotheses. The latter, consisting in a piecewise linearization of the microscopic displacement field, impose first that the displacement discontinuity vectors across the debonded interfaces are necessarily affine functions of spatial coordinates. Since an affine function can not be equal to zero on a segment and different from zero elsewhere, there is either (total) decohesion almost everywhere or there is no decohesion. This means that the simplified (piece-wise linear) kinematics put forward by Christoffersen (1983) does not allow one to account for partial decohesion of grain/matrix interfaces. Moreover, the hypothesis stipulating the identical displacement gradient $\mathbf{f}^0$ for opposite grains separated by a given layer imposes either no decohesion, or simultaneous decohesion of its both interfaces. Physically speaking, for grains of different size and, in particular, two opposite interfaces of different geometry and area, it is clear that a single crack along one of the interfaces (one-sided decohesion) would be more realistic than two simultaneous events. Unfortunately, the kinematics framework of the Christoffersen pattern does not allow for such one-sided local decohesion. So, in order to make the double decohesion acceptable, one completes the geometrical basis of the Christoffersen theory by adding the following assumption:

- any two opposite interfaces are supposed to have comparable geometrical properties (shape and area).

Since two opposite interfaces remain parallel during motion, such an assumption regarding their geometry gives some physical justification to the fact that when the first one is debonded, the second is too. We are aware that the latter assumption, by adding a supplementary constraint to the schematization, leads to restriction of the class of particulate composite microstructures that could be modeled with Christoffersen's original geometrical scheme. Nevertheless, it seems to be a necessary compromise to legitimate the Christoffersen kinematical framework in the presence of damage.

Having supposed the above simplification and considering the parallelism of interfaces in the course of deformation, it is reasonable to consider that the mean displacement discontinuity vectors across the interfaces I_1^α and I_2^α of the debonded layer α defined by

$$\begin{aligned}\langle b_i^{\alpha 1}\rangle_{I_1^\alpha} &= f_{ij}^{\alpha D} y_j^{B1} + c_i^{\alpha D1} = b_i^{\alpha 1}(\boldsymbol{y}^{B1})\\ \langle b_i^{\alpha 2}\rangle_{I_2^\alpha} &= f_{ij}^{\alpha D} y_j^{B2} + c_i^{\alpha D2} = b_i^{\alpha 2}(\boldsymbol{y}^{B2})\end{aligned} \tag{2.14}$$

are opposite. B_1 and B_2 are the centres of the interfaces I_1^α and I_2^α, respectively.

In the following, one attempts to simplify expression (2.13) obtained for $\mathbf{f}^\alpha$ involving for instance the terms $\mathbf{f}^{\alpha D}$, $\boldsymbol{c}^{\alpha D1}$ and $\boldsymbol{c}^{\alpha D2}$ considered as data characterizing microcracks at the boundaries of the debonded layer α considered. The relevant motivation is the advantage of reducing the number of entities that will characterize the effects of microcracks inside the RVE in the expressions further obtained for local fields and the homogenized stress-strain relation. To this aim, one introduces now the following assumption concerning the vectors $\boldsymbol{c}^{\alpha D1}$ and $\boldsymbol{c}^{\alpha D2}$, for which – it should be emphasized – no condition has been imposed by the Christoffersen methodology:

- The contribution of constant vectors $\boldsymbol{c}^{\alpha D1}$ and $\boldsymbol{c}^{\alpha D2}$ in displacement jumps $b_i^{\alpha 1}(\boldsymbol{y}) = f_{ij}^{\alpha D} y_j + c_i^{\alpha D1}$ and $b_i^{\alpha 2}(\boldsymbol{y}) = f_{ij}^{\alpha D} y_j + c_i^{\alpha D2}$ across I_1^α and I_2^α, respectively, are considered negligible (i.e. null).

The latter hypothesis consists in fact in choosing particular, simple and linear forms of displacement jumps considered as data of the local problem.

General forms (2.5) and (2.11) are thus replaced (with moreover $\mathbf{f}^{\alpha D} \equiv \mathbf{f}^{\alpha D1} = \mathbf{f}^{\alpha D2}$) by

$$b_i^{\alpha 1}(\boldsymbol{y}) = f_{ij}^{\alpha D} y_j \qquad\qquad b_i^{\alpha 2}(\boldsymbol{y}) = f_{ij}^{\alpha D} y_j \tag{2.15}$$

In this way, the displacement gradient $\mathbf{f}^{\alpha}$ for a debonded layer α takes the simplified expression

$$f_{ij}^{\alpha} = f_{ij}^{0} + (F_{ik} - f_{ik}^{0}) d_k^{\alpha} \frac{n_j^{\alpha}}{h^{\alpha}} + f_{ij}^{\alpha D} \tag{2.16}$$

Moreover, the following simple relationship exists now between $\langle \boldsymbol{b}^{\alpha 1} \rangle_{I_1^{\alpha}} = -\langle \boldsymbol{b}^{\alpha 2} \rangle_{I_2^{\alpha}}$ and the unique term $\mathbf{f}^{\alpha D}$ representing the two microcracks effect on $\mathbf{f}^{\alpha}$ by subtracting $(2.14)_2$ from $(2.14)_1$ and suppressing the contribution of $\boldsymbol{c}^{\alpha D1}$ and $\boldsymbol{c}^{\alpha D2}$)

$$\langle b_i^{\alpha 1} \rangle_{I_1^{\alpha}} = -\langle b_i^{\alpha 2} \rangle_{I_2^{\alpha}} = -\frac{1}{2} f_{ij}^{\alpha D} c_j^{\alpha} \qquad\qquad c_j^{\alpha} = y_j^{B2} - y_j^{B1} \tag{2.17}$$

In (2.17), $\boldsymbol{c}^{\alpha}$ designates the vector connecting the centres B_1 and B_2 of two opposite interfaces (see Fig. 2).

The displacement gradient $\mathbf{f}^{\alpha}$ for any layer α whose both interfaces are cohesive, obtained by using the continuity of displacements on the grain/layer interfaces according to the Christoffersen methodology, remains given by (2.4)

$$f_{ij}^{\alpha} = f_{ij}^{0} + (F_{ik} - f_{ik}^{0}) d_k^{\alpha} \frac{n_j^{\alpha}}{h^{\alpha}} \tag{2.18}$$

In view of (2.16) for a debonded layer and (2.18) for a cohesive one, the strain as well as rotation is controlled by $\mathbf{F}$, the macroscopic displacement gradient, $\mathbf{f}^0$, the grain displacement gradient, but also by geometrical features of the layer α under consideration. One may emphasize the physical relevance of such a dependence on local morphological parameters: it allows one to account for the microstructure effect on deformation mechanisms of the matrix. In this way, the Christoffersen kinematical framework offers a way to take into account some strain heterogeneity in the matrix phase in the homogenized behaviour estimation. It is stressed that taking into account field fluctuations in phases represents actually a crucial challenge in micromechanics especially for non linear and/or time-dependent behaviour (see e.g. Ponte Castañeda, 2002; Moulinec et Suquet, 2003). It is to be noted that the strain heterogeneity in the matrix is also influenced by damage via the dependence of $\mathbf{f}^{\alpha}$ on $\mathbf{f}^{\alpha D}$ (for debonded layers). At last, due to the assumption neglecting the description of complex effects in interlayer zones (see surrounded zones in Fig. 1), each layer is in the Christoffersen framework subjected to loading uniquely via its adjacent grains. In this way, there is no direct interaction between layers; the transmission through the grains-and-layers assembly strongly involves the grains as expressed through the presence of $\mathbf{f}^0$ in (2.16) and (2.18).

2.2.2. *Micro-macro relations*

The focus is here on establishing micro–macro relationships essential for the ultimate solution to the local problem, i.e. for determination of the unknown $\mathbf{f}^0$ according to the procedure outlined further. In the same spirit as in the kinematic description, one follows step-by-step – in the presence of damage – the corresponding method by Christoffersen (1983) for the sound material.

In order to ensure compatibility between local motion in accordance with the above kinematical description and global motion characterized by $\mathbf{F}$, the following average relation, the counterpart of relation (2.13) in Christoffersen (1983), including now the contribution of material discontinuities, is imposed

$$F_{ij} = (1-c)f_{ij}^0 + \frac{1}{V}\sum_{\alpha} f_{ij}^{\alpha} A^{\alpha} h^{\alpha} + \frac{1}{V}\sum_{k}\Big(\int\limits_{I_1^k} b_i^{k1} n_j^k \, da - \int\limits_{I_2^k} b_i^{k2} n_j^k \, da\Big) \tag{2.19}$$

where V represents the volume of grains and layers, A^{α} is the projected area of the αth layer and $c = V^{-1}\sum_{\alpha} A^{\alpha} h^{\alpha}$ is the ratio of the layer volume to the volume V. The subscripts α, k under summation symbols designate summations over all layers contained in the RVE and over layers with debonded interfaces, respectively. After some manipulations using (2.18), and (2.16), for $\mathbf{f}^{\alpha}$ for the layers α whose both interfaces are cohesive, respectively debonded, and (2.15) to express $\boldsymbol{b}^{k1}$ and $\boldsymbol{b}^{k2}$, one may prove that the geometrical condition established by Christoffersen for the sound material, namely

$$\frac{1}{V}\sum_{\alpha} d_i^{\alpha} n_j^{\alpha} A^{\alpha} = \delta_{ij} \tag{2.20}$$

remains necessary in the presence of damage to ensure the compatibility between local and global motions, i.e. relationship (2.19). In (2.20) δ_{ij} is the Kronecker's symbol. In the work by Christoffersen (1983), geometrical condition (2.20) related to the composite morphology may be seen as a discriminating criterion of applicability for the Christoffersen-type approach. Thus, it seems coherent to retrieve such a condition in the presence of interfacial damage (cracks).

The principle of macro-homogeneity for the RVE subjected to uniform tractions is given by Christoffersen (1983) – see Eq. (3.1) in the reference quoted. The corresponding expression extended here and accounting for interface discontinuities takes the following form

$$
\begin{aligned}
\Sigma_{ij}F_{ji} = (1-c)\sigma^0_{ij}f^0_{ji} + \frac{1}{V}\sum_\alpha \sigma^\alpha_{ij}f^\alpha_{ji}A^\alpha h^\alpha + \\
+\frac{1}{V}\sum_k\Big(\int\limits_{I^k_1}\sigma_{ij}n^k_j b^{k1}_i\,da - \int\limits_{I^k_2}\sigma_{ij}n^k_j b^{k2}_i\,da\Big)
\end{aligned}
\tag{2.21}
$$

for any arbitrary F and f^0 and any stress field $\boldsymbol{\sigma}$, statically admissible with the macroscopic stress $\boldsymbol{\Sigma}$. $\boldsymbol{\sigma}^0$ and $\boldsymbol{\sigma}^\alpha$ represent average stresses in the grains and in the αth layer, respectively. After some manipulations using (2.16) and (2.18) to express f^α, (2.15) for $\boldsymbol{b}^{k1}$ and $\boldsymbol{b}^{k2}$, and taking successively two particular values for f^0, namely $\mathsf{f}^0 = \mathsf{F}$ and $\mathsf{f}^0 = \mathbf{0}$ as it was done for the sound material, it can be shown from (2.21) that the system established by Christoffersen

$$
\left\{
\begin{array}{ll}
\Sigma_{ij} = \langle\sigma_{ij}\rangle_V = (1-c)\sigma^0_{ij} + \dfrac{1}{V}\displaystyle\sum_\alpha \sigma^\alpha_{ij}A^\alpha h^\alpha & \\
\Sigma_{ij} = \dfrac{1}{V}\displaystyle\sum_\alpha t^\alpha_i d^\alpha_j = \dfrac{1}{V}\displaystyle\sum_\alpha t^\alpha_j d^\alpha_i & t^\alpha_j = \sigma^\alpha_{kj}n^\alpha_k A^\alpha
\end{array}
\right.
\tag{2.22}
$$

remains valid in the presence of damage. In (2.22), $\boldsymbol{t}^\alpha$ represents the total force transmitted through the interfacial layer. Note that, although the first averaging is "classically" exploited in the micromechanics, the second one remains specific to the Christoffersen-type approach: stresses are seen from a granular viewpoint as forces transmitted from grain to grain by layers acting as contacts zones. For the debonded layer α, two cases must be considered. When cracks located at its boundaries are open (i.e $\langle b^{\alpha 1}_i\rangle_{I^\alpha_1} n^\alpha_i = -\langle b^{\alpha 2}_i\rangle_{I^\alpha_2} n^\alpha_i > 0$) then $\boldsymbol{t}^\alpha = \mathbf{0}$. When they are closed (i.e $\langle b^{\alpha 1}_i\rangle_{I^\alpha_1} n^\alpha_i = -\langle b^{\alpha 2}_i\rangle_{I^\alpha_2} n^\alpha_i = 0$) and in the framework of this exploratory study, it is supposed that no sliding is allowed, so that $\boldsymbol{t}^\alpha$ is integrally transmitted. For a cohesive layer α, $\boldsymbol{t}^\alpha$ is considered as fully conveyed as it was in the case of all layers in the absence of damage.

According to the Christoffersen methodology, the following consists in searching f^0 in such a way that the real stress field, namely this associated to the strain field by local constitutive laws, satisfies system (2.22).

3. Application to viscoelastic composite materials

The class of heterogeneous materials considered is that of particulate composite materials which can be considered as composed of isotropic linear-elastic grains embedded in a viscoelastic matrix (see Nadot-Martin *et al.*, 2003). At first, mean features of the matrix viscoelastic law are recalled. This constitutes a preliminary step before going on to find $\mathbf{f}^0$ and to establish the full set of localization relations, as it was done for the sound material by Nadot-Martin *et al.* (2003), but here is done in the presence of damage by grain/matrix debonding. Then, the macroscopic homogenized stress is derived from $(2.22)_1$. Finally, a discussion is presented in order to quantify the damage influence on the local and global scale levels.

3.1. Viscoelastic law for the matrix

The matrix occupying each elementary layer α is considered as viscoelastic and isotropic according to the thermodynamically consistent internal variable representation given by Nadot-Martin *et al.* (2003). The dissipative process related to viscoelastic relaxation is accounted for via the symmetric, strain-like, tensorial internal variable $\boldsymbol{\gamma}$. The free energy per unit volume and correspondingly the total stress are decomposed into two terms, a reversible function of the total strain $\boldsymbol{\varepsilon}$, and a viscous function of $\boldsymbol{\gamma}$. The reversible and viscous stresses are obtained by partial derivation of the free energy with respect to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\gamma}$. The evolution of $\boldsymbol{\gamma}$ which can be interpreted as inelastic-viscous or otherwise as 'delayed elastic' strain is given by law $(3.3)_1$ employing, for simplicity, a single relaxation time τ

$$w(\boldsymbol{\varepsilon},\boldsymbol{\gamma}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{L}^{(e)\ell} : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\gamma} : \mathbf{L}^{(v)} : \boldsymbol{\gamma} \tag{3.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(r)} + \boldsymbol{\sigma}^{(v)} = \mathbf{L}^{(e)\ell} : \boldsymbol{\varepsilon} + \mathbf{L}^{(v)} : \boldsymbol{\gamma} \tag{3.2}$$

$$\begin{aligned} &\dot{\boldsymbol{\gamma}} + \frac{1}{\tau}\boldsymbol{\gamma} = \dot{\boldsymbol{\varepsilon}} \qquad\qquad \boldsymbol{\gamma}(t=0) = \mathbf{0} \\ &d^{(v)} = \boldsymbol{\sigma}^{(v)} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\gamma}}) = \frac{1}{\tau}\boldsymbol{\gamma} : \mathbf{L}^{(v)} : \boldsymbol{\gamma} \geqslant 0 \end{aligned} \tag{3.3}$$

$\mathbf{L}^{(e)\ell}$ and $\mathbf{L}^{(v)}$ are fourth-order tensors of the elastic and viscous moduli for the matrix.

3.2. Solution to the local problem and expression of homogenized stress

The purpose is to resolve system (2.22) in order to get $\mathbf{f}^0$ by considering grains as isotropic linear-elastic and the matrix layers as viscoelastic according to the model presented in Subsection 3.1. All the grains have here identical moduli denoted by $\mathbf{L}^0$. Mechanical properties of the matrix (moduli $\mathbf{L}^{(e)\ell}$, $\mathbf{L}^{(v)}$ and relaxation characteristic τ) are considered as homogeneous, namely the same for all layers. Consequently, as $\boldsymbol{\varepsilon}^\alpha = \mathrm{Sym}\,\mathbf{f}^\alpha$ is uniform over the αth layer (see kinematical assumption 3 in Subsection 2.2.1), the corresponding viscoelastic relaxation $\boldsymbol{\gamma}$ introduced by $(3.3)_1$ is also uniform for a given relaxation state; it is denoted by $\boldsymbol{\gamma}^\alpha$. It is also the case for all thermodynamic quantities involved in the matrix model.

From a methodological viewpoint, calculations to determine $\mathbf{f}^0$ from (2.22) in the presence of damage are similar to those required for the sound material (see for comparison Subsection 3.2 in Nadot-Martin *et al.*, 2003). Nevertheless, it is to be recalled that the summation in $(2.22)_2$ is here to be considered over layers either cohesive or with closed cracks. One begins by inserting the microscopic laws formulated in terms of displacement gradients rather than in terms of strain in system (2.22). Then, (2.18) is substituted for $\mathbf{f}^\alpha$ for the layers α whose both interfaces are cohesive, while (2.16) is put for the layers debonded. Finally, by using geometrical condition (2.20) and eliminating Σ_{ij} between both equations of (2.22), one obtains the form of $\mathbf{f}^0$ relevant to the local problem in the presence of damage as follows

$$
\begin{aligned}
f^0_{ij} = {} & \underbrace{(Id^1 - B'^{-1} : A')_{ijkl} F_{lk}}_{f^{0(r)}_{ij}} + \\
& \underbrace{-B'^{-1}_{ijuv} L^{(v)}_{mukl} \Big(\frac{1}{V} \sum_{\alpha'} \Pi^{\alpha'}_{vm} \gamma^{\alpha'}_{lk} A^{\alpha'} h^{\alpha'} + \delta_{vm} \frac{1}{V} \sum_{\beta} \gamma^{\beta}_{lk} A^\beta h^\beta \Big)}_{f^{0(v)}_{ij}} + \\
& \underbrace{-B'^{-1}_{ijuv} L^{(e)\ell}_{mukl} \Big(\frac{1}{V} \sum_{f} \Pi^{f}_{vm} \varepsilon^{fD}_{lk} A^{f} h^{f} + \delta_{vm} \frac{1}{V} \sum_{\beta} \varepsilon^{\beta D}_{lk} A^\beta h^\beta \Big)}_{f^{0(d)}_{ij}}
\end{aligned}
\tag{3.4}
$$

with, for any layer α, $\mathbf{\Pi}^\alpha = \boldsymbol{\delta} - \boldsymbol{d}^\alpha \otimes \boldsymbol{n}^\alpha / h^\alpha$ and where the tensors $\mathbf{A}'$, $\mathbf{B}'$ degraded by the presence of damage are defined as follows

$$
A'_{ijkl} = \langle L^{(e)}_{ijkl} \rangle_V - L^{(e)\ell}_{mjkl}(\delta_{im} - D_{im}) \qquad A_{ijkl} = \langle L^{(e)}_{ijkl} \rangle_V - L^{(e)\ell}_{ijkl} \tag{3.5}
$$

$$
B'_{ijkl} = A_{ijkl} - L^{(e)\ell}_{mjkl}(\delta_{im} - D_{im}) + L^{(e)\ell}_{mjnl}(\overline{T}_{imkn} - \overline{D}_{imkn}) \tag{3.6}
$$

$$
\overline{T}_{ijkl} = \frac{1}{V} \sum_\alpha d^\alpha_i n^\alpha_j d^\alpha_k n^\alpha_l \frac{A^\alpha}{h^\alpha} \tag{3.7}
$$

$$
D_{ij} = \frac{1}{V} \sum_\beta d^\beta_i n^\beta_j A^\beta \qquad \overline{D}_{ijkl} = \frac{1}{V} \sum_\beta d^\beta_i n^\beta_j d^\beta_k n^\beta_l \frac{A^\beta}{h^\beta} \tag{3.8}
$$

In the above relations, the subscripts α, α', β and f under summation symbols denote summations over all layers, layers either cohesive or with closed cracks, layers with open cracks only and layers with closed cracks only. In (3.4), $\boldsymbol{\varepsilon}^{\beta D} = \mathrm{Sym}\,\mathbf{f}^{\beta D}$, $\boldsymbol{\varepsilon}^{fD} = \mathrm{Sym}\,\mathbf{f}^{fD}$ and one has assumed invertibility of $\mathbf{B}'$ with respect to the identity tensor $\mathbf{Id}^1$ defined by $Id^1_{ijkl} = \delta_{il}\delta_{jk}$. The form of (3.4) represents a remarkable decomposition into a reversible term $\mathbf{f}^{0(r)}$, depending linearly on the macroscopic gradient $\mathbf{F}$, a viscous one $\mathbf{f}^{0(v)}$, function of variables $\boldsymbol{\gamma}^\alpha$ for $\alpha = 1, \ldots, N$ – with N being the total number of layers inside the RVE – and a damage-induced one $\mathbf{f}^{0(d)}$ involving the full set $\{\boldsymbol{\varepsilon}^{kD}\} = \{\boldsymbol{\varepsilon}^{fD}\} \cup \{\boldsymbol{\varepsilon}^{\beta D}\}$ related to the effect of any kind of cracks (closed and open) inside the RVE. These three contributions depend on the damage state through the tensors $\mathbf{D}$ and $\overline{\mathbf{D}}$ (see $\mathbf{A}'$, $\mathbf{B}'$). The same can be done for $\mathbf{f}^\alpha$ after employing (2.18) and (2.16) for a cohesive and debonded layer, respectively . At last, the local strain field with respect to $\boldsymbol{y}$ in the grains and matrix layers is obtained in the following additive form

$$\boldsymbol{\varepsilon}(\boldsymbol{y}) = \mathbf{C}(\boldsymbol{y}) : \mathbf{E} + \boldsymbol{\varepsilon}^{(v)}(\boldsymbol{y}) + \boldsymbol{\varepsilon}^{(d)}(\boldsymbol{y}) + \begin{cases} \boldsymbol{\varepsilon}^{\alpha D} & \text{for } \boldsymbol{y} \in \text{ debonded layer } \alpha \\ \mathbf{0} & \text{elsewhere} \end{cases} \tag{3.9}$$

$$C_{ijkl}(\boldsymbol{y}) = \begin{cases} C^0_{ijkl}(\mathbf{D}, \overline{\mathbf{D}}) = (Id - Id : B'^{-1} : A')_{ijkl} & \text{for } \boldsymbol{y} \in \text{ grains} \\ C^\alpha_{ijkl}(\mathbf{D}, \overline{\mathbf{D}}) = Id_{ijkl} + & \\ \quad -Id_{ijuv}(B'^{-1} : A')_{vmkl}\mathit{\Pi}^\alpha_{mu} & \text{for } \boldsymbol{y} \in \text{ layer } \alpha,\ \forall\alpha \end{cases} \tag{3.10}$$

$$\varepsilon^{(v)}_{ij}(\boldsymbol{y}) = \begin{cases} \varepsilon^{0(v)}_{ij}(\{\boldsymbol{\gamma}^\alpha\}, \mathbf{D}, \overline{\mathbf{D}}) = Id_{ijkl} f^{0(v)}_{lk} & \text{for } \boldsymbol{y} \in \text{ grains} \\ \varepsilon^{\alpha(v)}_{ij}(\{\boldsymbol{\gamma}^\alpha\}, \mathbf{D}, \overline{\mathbf{D}}) = Id_{ijuv} f^{0(v)}_{vm}\mathit{\Pi}^\alpha_{mu} & \text{for } \boldsymbol{y} \in \text{ layer } \alpha,\ \forall\alpha \end{cases} \tag{3.11}$$

$$\varepsilon^{(d)}_{ij}(\boldsymbol{y}) = \begin{cases} \varepsilon^{0(d)}_{ij}(\{\boldsymbol{\varepsilon}^{kD}\}, \mathbf{D}, \overline{\mathbf{D}}) = Id_{ijkl} f^{0(d)}_{lk} & \text{for } \boldsymbol{y} \in \text{ grains} \\ \varepsilon^{\alpha(d)}_{ij}(\{\boldsymbol{\varepsilon}^{kD}\}, \mathbf{D}, \overline{\mathbf{D}}) = Id_{ijuv} f^{0(d)}_{vm}\mathit{\Pi}^\alpha_{mu} & \text{for } \boldsymbol{y} \in \text{ layer } \alpha,\ \forall\alpha \end{cases} \tag{3.12}$$

with $\mathbf{E} = \mathrm{Sym}\,\mathbf{F}$. As expected, the degraded elastic strain concentration tensor satisfies $\langle \mathbf{C} \rangle_V = \mathbf{Id}$ – with $\mathbf{Id}$ being the classical fourth-order identity tensor defined by $Id_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$ – and the fields $\boldsymbol{\varepsilon}^{(v)}$ and $\boldsymbol{\varepsilon}^{(d)}$ the properties $\langle \boldsymbol{\varepsilon}^{(v)} \rangle_V = \mathbf{0}$ and $\langle \boldsymbol{\varepsilon}^{(d)} \rangle_V = \mathbf{0}$, respectively. At last, the overall (average) stress is derived from $(2.22)_1$

$$\boldsymbol{\Sigma} = \mathbf{L}(\mathbf{D}, \overline{\mathbf{D}}) : \mathbf{E} + \boldsymbol{\Sigma}^{(v)}(\{\boldsymbol{\gamma}^\alpha\}, \mathbf{D}, \overline{\mathbf{D}}) + \boldsymbol{\Sigma}^{(d)}(\{\boldsymbol{\varepsilon}^{kD}\}, \mathbf{D}, \overline{\mathbf{D}}) \tag{3.13}$$

$$\mathbf{L}(\mathbf{D}, \overline{\mathbf{D}}) = \langle \mathbf{L}^{(e)} \rangle_V - \mathbf{A} : \mathbf{B}'^{-1} : \mathbf{A}' \tag{3.14}$$

$$\mathbf{\Sigma}^{(v)} = \mathbf{A} : \mathbf{f}^{0(v)}(\{\gamma^\alpha\}, \mathbf{D}, \overline{\mathbf{D}}) + \mathbf{L}^{(v)} : \frac{1}{V}\sum_\alpha \gamma^\alpha A^\alpha h^\alpha \tag{3.15}$$

$$\mathbf{\Sigma}^{(d)} = \mathbf{A} : \mathbf{f}^{0(d)}(\{\varepsilon^{kD}\}, \mathbf{D}, \overline{\mathbf{D}}) + \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_k \varepsilon^{kD} A^k h^k \tag{3.16}$$

3.3. Discussion

In order to discuss the forms of results on micro and macro levels in the presence of damage, it may be convenient to compare them with those obtained by Nadot-Martin *et al.* (2003) for the sound material.

Table 1. Localization results and expression of the homogenized stress for the sound material (Nadot-Martin *et al.*, 2003)

<u>Local strain field:</u>

$$\boldsymbol{\varepsilon}(\boldsymbol{y}) = \mathbf{C}(\boldsymbol{y}) : \mathbf{E} + \boldsymbol{\varepsilon}^{(v)}(\boldsymbol{y})$$

$$C_{ijkl}(\boldsymbol{y}) = \begin{cases} C^0_{ijkl} = (Id - Id : B^{-1} : A)_{ijkl} & \text{for } \boldsymbol{y} \in \text{grains} \\ C^\alpha_{ijkl} = Id_{ijkl} - Id_{ijuv}(B^{-1} : A)_{vmkl}\Pi^\alpha_{mu} & \text{for } \boldsymbol{y} \in \text{layer } \alpha, \forall\alpha \end{cases}$$

$$\varepsilon^{(v)}_{ij}(\boldsymbol{y}) = \begin{cases} \varepsilon^{0(v)}_{ij}(\{\gamma^\alpha\}) = Id_{ijkl} f^{0(v)}_{lk} & \text{for } \boldsymbol{y} \in \text{grains} \\ \varepsilon^{\alpha(v)}_{ij}(\{\gamma^\alpha\}) = Id_{ijuv} f^{0(v)}_{vm} \Pi^\alpha_{mu} & \text{for } \boldsymbol{y} \in \text{layer } \alpha,\ \forall\alpha \end{cases}$$

<u>Homogenized stress:</u>

$$\mathbf{\Sigma} = \mathbf{L} : \mathbf{E} + \mathbf{\Sigma}^{(v)}(\{\gamma^\alpha\})$$

$$\mathbf{L} = \langle \mathbf{L}^{(e)} \rangle_V - \mathbf{A} : \mathbf{B}^{-1} : \mathbf{A}$$

$$\mathbf{\Sigma}^{(v)} = \mathbf{A} : \mathbf{f}^{0(v)}(\{\gamma^\alpha\}) + \mathbf{L}^{(v)} : \frac{1}{V}\sum_\alpha \gamma^\alpha A^\alpha h^\alpha$$

....

with:

$$\mathbf{A} = \langle \mathbf{L}^{(e)} \rangle_V - \mathbf{L}^{(e)\ell}$$

$$B_{ijkl} = A_{ijkl} - L^{(e)\ell}_{ijkl} + L^{(e)\ell}_{mjnl}\overline{T}_{imkn}$$

$$\overline{T}_{ijkl} = \frac{1}{V}\sum_\alpha d^\alpha_i n^\alpha_j d^\alpha_k n^\alpha_l \frac{A^\alpha}{h^\alpha}$$

$$f^{0(v)}_{ij}(\{\gamma^\alpha\}) = -B^{-1}_{ijuv} L^{(v)}_{mukl} \frac{1}{V}\sum_\alpha \Pi^\alpha_{vm}\gamma^\alpha_{lk} A^\alpha h^\alpha$$

At the local level, one may observe that the degraded elastic concentration tensor given by (3.10) has the same form as $\mathbf{C}$ for the sound material with $\mathbf{A}'$

and $\mathbf{B}'$ replacing $\mathbf{A}$ and $\mathbf{B}$. Moreover, the strain field (3.9) for any point in grains or layers depends through the term $\boldsymbol{\varepsilon}^{(v)}$ on the full set of relaxations $\{\boldsymbol{\gamma}^\alpha\}$. The internal variable $\boldsymbol{\gamma}^\alpha$ representing memory of the αth layer clearly indicates viscoelastic interactions between the full set of matrix layers and the set of grains in the RVE. In the same manner as for the sound material, this dependence directly results from the term $\mathbf{f}^{0(v)}$ which, by means of (2.16) and (2.18) appears in the expression of $\mathbf{f}^\alpha$ and, therefore, in that of the strain field, see (3.11). Nevertheless, one may note that the more complex structure of $\mathbf{f}^{0(v)}$ (see (3.4)) in the presence of damage shows that the damage tends to enhance the complexity of viscoelastic interferences taken into account. Moreover, the strain field (3.9) for any point in grains or layers (cohesive or not) depends on damage through the tensors $\mathbf{D}$ and $\overline{\mathbf{D}}$ (appearing in $\mathbf{A}'$ and $\mathbf{B}'$) but also on the term $\boldsymbol{\varepsilon}^{(d)}$ depending on the full set $\{\boldsymbol{\varepsilon}^{kD}\} = \{\boldsymbol{\varepsilon}^{\beta D}\} \cup \{\boldsymbol{\varepsilon}^{fD}\}$ related to the effect of any kind of cracks (open or closed) inside the RVE. The latter dependence results from the term $\mathbf{f}^{0(d)}$ which, for the same reasons as $\mathbf{f}^{0(v)}$, appears in the expression of the strain field, see (3.12). This is not surprising when reported to comments formulated at the end of Subsection 2.2.1 concerning the transmission inside the aggregate. In particular, for the debonded layer α, one may distinguish two kinds of contribution of damage to the corresponding "overall" strain in the layer: a "local" one, $\boldsymbol{\varepsilon}^{\alpha D}$, related to microcracks located at its own boundaries (its "own" defects) and a "non-local" one, $\boldsymbol{\varepsilon}^{\alpha(d)}$, involving the effect (via $\mathbf{f}^{0(d)}$) of the whole set of microcracks inside the RVE, in other words the effect of microcracks at the interfaces of other layers in addition to the influence of those at its own boundaries.

At the global level, the overall stress given by (3.13) is split into a reversible part and a viscous one influenced by damage through $\mathbf{D}$ and $\overline{\mathbf{D}}$ and completed (when compared to that for the sound material) by the damage-induced stress $\boldsymbol{\Sigma}^{(d)}$. Note that the forms of viscous stress for the sound and damaged materials are the same, the difference is in the detailed expression of $\mathbf{f}^{0(v)}$ relevant to local viscoelastic interactions depending here on the damage state in addition to the set $\{\boldsymbol{\gamma}^\alpha\}$ acquiring the status of macroscopic viscoelastic internal variables. The first terms of (3.15) and (3.16), namely $\mathbf{A} : \mathbf{f}^{0(v)}$ and $\mathbf{A} : \mathbf{f}^{0(d)}$, correspond respectively to the macroscopic consequences of viscoelastic interactions and "non-local" damage effects.

It can be seen that $\mathbf{A}'$, $\mathbf{B}'$ and therefore $\mathbf{C}$ and $\mathbf{L}(\mathbf{D}, \overline{\mathbf{D}})$, are degraded only by open cracks via $\mathbf{D}$ and $\overline{\mathbf{D}}$, see (3.8). This is due to the assumption of no sliding on closed crack lips (infinite friction coefficient). Being tensorial by nature, $\mathbf{D}$ and $\overline{\mathbf{D}}$ allow one to account for the damage induced anisotropy. By depending on the vectors $\boldsymbol{d}^\beta$ and not only on the crack normal vectors $\boldsymbol{n}^\beta$, the damage tensors $\mathbf{D}$ and $\overline{\mathbf{D}}$ emerging from the present morphology-based modelling take into account the granular character of the composite microstructure considered. Moreover, since $\mathbf{D}$ is not symmetric, the damage induced anisotropy may be very complex. It is stressed that the scale transition at stake accounts also for the initial morphology and internal organization of constituents through the presence of the fourth-order structural tensor $\overline{\mathbf{T}}$ given by (3.7) in the local and homogenized expressions (via $\mathbf{B}'$ for the damaged material and $\mathbf{B}$ for the sound one). The reader may refer to Christoffersen (1983), where it is shown that $\overline{\mathbf{T}}$ reflects material texture and irregularities in the grain shape and in the layer thickness. In this way, the Christoffersen-type approach extension in the presence of damage, applied here to a viscoelastic composite, allows one to take into account, in a general 3D context, coupling effects between the primary anisotropy, if any, (via $\overline{\mathbf{T}}$) and the secondary, damage-induced one (via $\mathbf{D}$ and $\overline{\mathbf{D}}$).

At last, when the number of open cracks is equal to zero, i.e. for exclusively closed cracks, the reversible part of the local strain field and, furthermore, the homogenized reversible moduli become equal to those of the sound material. The viscous part $\boldsymbol{\varepsilon}^{(v)}$ of the local strain field becomes equal to that of the sound material as well as $\mathbf{\Sigma}^{(v)}$ at the macroscopic scale. The term $\boldsymbol{\varepsilon}^{(d)}$, depending only on $\{\boldsymbol{\varepsilon}^{fD}\}$, accounts for the blockage effect of closed cracks inside the RVE as the corresponding macroscopic damage-induced stress $\mathbf{\Sigma}^{(d)}$. Thus, the modelling is potentially capable of describing unilateral effects. In the limit case where there is no crack, the local and global responses are identical to those obtained for the sound material. This principal backwards confrontation shows that the methodological coherence is being preserved between the sound and damaged composites.

As macroscopic state variables, one has already mentioned the whole set $\{\gamma^\alpha\}$ accounting for the relaxation state of the composite. Homogenized stress (3.13) conveys also a full set $\{\boldsymbol{\varepsilon}^{kD}\} = \{\boldsymbol{\varepsilon}^{\beta D}\} \cup \{\boldsymbol{\varepsilon}^{fD}\}$. Let examine now the status of $\{\boldsymbol{\varepsilon}^{kD}\}$. Remarking, according to (3.4), that

$$
\begin{aligned}
\mathbf{f}^{0(d)}(\{\boldsymbol{\varepsilon}^{kD}\},\mathbf{D},\overline{\mathbf{D}}) = {} & \underbrace{-(B'^{-1} : L^{(e)\ell})_{ijkl}\frac{1}{V}\sum_{\beta}\varepsilon^{\beta D}_{lk}A^\beta h^\beta +}_{\mathbf{f}^{0(d)1}(\{\boldsymbol{\varepsilon}^{\beta D}\},\mathbf{D},\overline{\mathbf{D}})} \\
& \underbrace{-B'^{-1}_{ijuv} : L^{(e)\ell}_{mukl}\frac{1}{V}\sum_{f}\Pi^f_{vm}\varepsilon^{fD}_{lk}A^f h^f}_{\mathbf{f}^{0(d)2}(\{\boldsymbol{\varepsilon}^{fD}\},\mathbf{D},\overline{\mathbf{D}})}
\end{aligned}
\tag{3.17}
$$

one may discern that the respective contributions in the damage-induced stress $\mathbf{\Sigma}^{(d)}$ of open and closed cracks are clearly additive. Indeed, when detailing somewhat (3.16) on the basis of partition (3.17), one obtains

$$\mathbf{\Sigma}^{(d)} = \underbrace{\mathbf{A} : \mathbf{f}^{0(d)1} + \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_{\beta} \boldsymbol{\varepsilon}^{\beta D} A^{\beta} h^{\beta}}_{\mathbf{\Sigma}^{(d)1}(\{\boldsymbol{\varepsilon}^{\beta D}\},\mathbf{D},\overline{\mathbf{D}})} + \underbrace{\mathbf{A} : \mathbf{f}^{0(d)2} + \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_{f} \boldsymbol{\varepsilon}^{f D} A^{f} h^{f}}_{\mathbf{\Sigma}^{(d)2}(\{\boldsymbol{\varepsilon}^{f D}\},\mathbf{D},\overline{\mathbf{D}})} \tag{3.18}$$

In (3.18), the set $\{\boldsymbol{\varepsilon}^{fD}\}$ acquires the status of macroscopic internal variables accounting for the distortion due to the blockage of closed cracks inside the RVE, and $\mathbf{\Sigma}^{(d)2}$ appears as the corresponding residual stress. At the microscopic level, $\boldsymbol{\varepsilon}^{\beta D}$ represents for a layer β the "local" contribution of open cracks located at its own boundaries to its total strain. It seems natural to think that the crack opening depends on the macroscopic strain $\mathbf{E}$ and therefore $\boldsymbol{\varepsilon}^{\beta D}$ as well. So, each $\boldsymbol{\varepsilon}^{\beta D}$ cannot *a priori* be considered as a macroscopic variable independent of $\mathbf{E}$. This is confirmed when noting that $\mathbf{L}(\mathbf{D},\overline{\mathbf{D}})$, given by (3.14), does not exhibit all symmetries required for the effective reversible moduli tensor suggesting that $\mathbf{\Sigma}^{(d)1}$ must depend, through $\{\boldsymbol{\varepsilon}^{\beta D}\}$, on $\mathbf{E}$. This remark shows that further analysis is necessary to explicit the dependence of each $\boldsymbol{\varepsilon}^{\beta D}$ on $\mathbf{E}$, that – via the term $\mathbf{\Sigma}^{(d)1}$ in the expression of $\mathbf{\Sigma}$ – will complete the linear part $\mathbf{L}(\mathbf{D},\overline{\mathbf{D}}) : \mathbf{E}$ of $\mathbf{\Sigma}$ and, therefore the form of reversible moduli. This is the aim of the next section where a complementary localization-homogenization procedure is advanced in order to express the local strain induced in a layer β by open cracks at its interfaces as a function of $\mathbf{E}$, $\mathbf{D}$, $\overline{\mathbf{D}}$ and local geometrical features of the layer concerned. In the spirit of a gradual, step-by-step approach to difficulties, this procedure is developed hereafter in the context of pure elasticity. It is a necessary and preliminary stage for further generalization in viscoelasticity.

4. A complementary localization-homogenization procedure for an elastic aggregate

4.1. Preliminaries

The purpose of this Section is to express $\boldsymbol{\varepsilon}^{\beta D}$ for an arbitrary layer β with open cracks at its own boundaries as a function of macroscopic state variables, the global strain $\mathbf{E}$ in particular. In the framework of the exploratory character of the approach advanced in this paper, the developments put forward below are performed in the elastic context, namely by considering the grains and the matrix (i.e. the set of layers) as linear elastic and isotropic.

The first step consists in the determination of the overall free-energy for the elastic heterogeneous material as the volume average of the local energy. After some calculations using the localization relations (see (3.9) to (3.12) where the viscous field $\boldsymbol{\varepsilon}^{(v)}$ is suppressed), employing geometrical statement (2.20) and the major symmetry of $\mathbf{B}$, the overall free energy is obtained in the following additive form

$$\begin{aligned} W = \Big\langle \frac{1}{2}\boldsymbol{\varepsilon} : \mathbf{L}^{(e)} : \boldsymbol{\varepsilon} \Big\rangle &= W^1(\mathbf{E},\mathbf{D},\overline{\mathbf{D}}) + W^2(\mathbf{E},\{\boldsymbol{\varepsilon}^{\beta D}\},\mathbf{D},\overline{\mathbf{D}}) + \\ &+ W^3(\mathbf{E},\{\boldsymbol{\varepsilon}^{fD}\},\mathbf{D},\overline{\mathbf{D}}) + W^4(\{\boldsymbol{\varepsilon}^{\beta D}\},\mathbf{D},\overline{\mathbf{D}}) + \\ &+ W^5(\{\boldsymbol{\varepsilon}^{\beta D}\},\{\boldsymbol{\varepsilon}^{fD}\},\mathbf{D},\overline{\mathbf{D}}) + W^6(\{\boldsymbol{\varepsilon}^{fD}\},\mathbf{D},\overline{\mathbf{D}}) \end{aligned} \tag{4.1}$$

where

$$W^1(\mathbf{E},\mathbf{D},\overline{\mathbf{D}}) = \frac{1}{2}\mathbf{E} : \langle {}^t\mathbf{C} : \mathbf{L}^{(e)} : \mathbf{C}\rangle_V : \mathbf{E} \tag{4.2}$$

$$\begin{aligned} \langle {}^t\mathbf{C} : \mathbf{L}^{(e)} : \mathbf{C}\rangle_V &= \langle \mathbf{L}^{(e)}\rangle_V + {}^t\mathbf{G} : \mathbf{B} : \mathbf{G} - [\mathbf{A} : \mathbf{G} + {}^t\mathbf{G} : \mathbf{A}] \\ \mathbf{G} &= \mathbf{B}'^{-1} : \mathbf{A}' \end{aligned} \tag{4.3}$$

and with W^i for $i = 2,\dots,6$ given in Appendix A. W^2 and W^3 are explicitly linear in $\mathbf{E}$ and linear with respect to each $\boldsymbol{\varepsilon}^{\beta D}$ and $\boldsymbol{\varepsilon}^{fD}$. W^5 depends linearly on each $\boldsymbol{\varepsilon}^{\beta D}$. The terms W^i for $i = 4,5,6$ do not depend explicitly on $\mathbf{E}$.

A quick comparison between the homogenized free energy and the expression of the global stress given by (3.13), where the viscous stress $\boldsymbol{\Sigma}^{(v)}$ is suppressed, shows immediately that the explicitly quadratic term in $\mathbf{E}$ of W, i.e. W^1, cannot give by derivation the linear term $\mathbf{L}(\mathbf{D},\overline{\mathbf{D}}) : \mathbf{E}$ of the stress except for only closed cracks inside the RVE. This provides a new confirmation of the dependence of each $\boldsymbol{\varepsilon}^{\beta D}$ on $\mathbf{E}$. A non-trivial problem consists then in quantifying the relationship between each $\boldsymbol{\varepsilon}^{\beta D}$ (for an arbitrary layer β) and $\mathbf{E}$. Note that the expression for $\boldsymbol{\varepsilon}^{\beta D}$ is not *a priori* postulated so that the strategy proposed can be viewed as a complementary "localization" procedure. To this aim, the thermodynamic framework is used as a guide.

From a thermodynamic viewpoint, the macroscopic stress must derive from the overall free energy with respect to $\mathbf{E}$. Consequently, the relation $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ has to be searched in such a way that the global elasticity law

$$\boldsymbol{\Sigma} = \frac{\partial W}{\partial \mathbf{E}} \tag{4.4}$$

be explicitly verified with $\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma}\rangle_V$ and $W = \langle w\rangle_V$ given by (3.13), with $\boldsymbol{\Sigma}^{(v)}$ suppressed, and by (4.1).

Since W^6 is independent of $\mathbf{E}$, $\partial W/\partial \mathbf{E} = \sum_{i=1}^{5} \partial W^i/\partial \mathbf{E}$. Assuming, for each layer β with open cracks at its boundaries, linearity of the relation $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ and its independence of the set $\{\boldsymbol{\varepsilon}^{fD}\}$, one can discern that: 1) W^2, W^4 and $\boldsymbol{\Sigma}^{(d)1}$ do not depend on $\{\boldsymbol{\varepsilon}^{fD}\}$; 2) W^3 and W^5, depending on the set $\{\boldsymbol{\varepsilon}^{fD}\}$, are linear functions of $\mathbf{E}$ (explicitly for W^3, implicitly through each $\boldsymbol{\varepsilon}^{\beta D}$ for W^5). Thus, for such a strategy $(\partial W^3/\partial \mathbf{E}) + (\partial W^5/\partial \mathbf{E})$ must correspond to the residual stress due to the blockage of closed crack lips. Precisely, one must have explicitly

$$\begin{aligned} &\frac{\partial W^3}{\partial \mathbf{E}} + \frac{\partial W^5}{\partial \mathbf{E}} = \boldsymbol{\Sigma}^{(d)2} \\ &\frac{\partial W^1}{\partial \mathbf{E}} + \frac{\partial W^2}{\partial \mathbf{E}} + \frac{\partial W^4}{\partial \mathbf{E}} = \mathbf{L}(\mathbf{D}, \overline{\mathbf{D}}) : \mathbf{E} + \boldsymbol{\Sigma}^{(d)1} \end{aligned} \tag{4.5}$$

In fact, while searching a relation between $\boldsymbol{\varepsilon}^{\beta D}$ and $\mathbf{E}$ directly (in order to assure that $\boldsymbol{\Sigma} = \partial W/\partial \mathbf{E} = \sum_{i=1}^{5} \partial W^i/\partial \mathbf{E}$ be satisfied) appears too complex when examining the detailed expressions of W^i for $i = 1, \ldots, 5$, it is easier to search it by satisfying $(4.5)_1$. It is stressed that ensuring $(4.5)_1$ is sufficient to ensure simultaneously $(4.5)_2$ and more generally (4.4). But the converse is not true. Indeed, searching a solution to $(4.5)_2$ would not be sufficient to ensure simultaneously (4.4) since the solution would not take into account the contribution of the set $\{\boldsymbol{\varepsilon}^{fD}\}$.

4.2. Solution

The aim is to find a linear relation $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ for each layer β with open cracks at its boundaries, satisfying equation $(4.5)_1$ with W^3, W^5 and $\boldsymbol{\Sigma}^{(d)2}$ given by (A.2), (A.4) and (3.18). From (A.2)-(A.4) and due to symmetries of $\mathbf{A}$, $\mathbf{L}^{(e)\ell}$ and $\mathbf{A}'$, the expression for $(\partial W^3/\partial \mathbf{E}) + (\partial W^5/\partial \mathbf{E})$ is obtained as follows

$$\begin{aligned} \frac{\partial W^3}{\partial E_{ij}} + \frac{\partial W^5}{\partial E_{ij}} &= A_{ijkl} f^{0(d)2}_{lk} + L^{(e)\ell}_{ijkl} \frac{1}{V} \sum_f \varepsilon^{fD}_{lk} A^f h^f + \\ &+ \Big\{ \Big[{}^t G_{ijuv} - {}^t\Big(\frac{\partial f^{0(d)1}_{uv}}{\partial E_{ij}} \Big) \Big] (B' - B)_{vukl} + L^{(e)\ell}_{vunl} \frac{1}{V} \sum_\beta \Pi^\beta_{kn} \frac{\partial \varepsilon^{\beta D}_{uv}}{\partial E_{ij}} A^\beta h^\beta \Big\} f^{0(d)2}_{lk} \end{aligned} \tag{4.6}$$

where one recognizes $\boldsymbol{\Sigma}^{(d)2}$ (see the first line) given by (3.18). So, $(4.5)_1$ is satisfied for any damage configuration, in particular for any $\mathbf{f}^{0(d)2}$, only if the term between braces in (4.6) is null. Moreover, by using the expressions for

$\mathbf{B}$ and $\mathbf{B}'$ (see Table 1 and (3.6), respectively) in order to develop $\mathbf{B}' - \mathbf{B}$, it follows

$$
\begin{aligned}
& L^{(e)\ell}_{vunl} \frac{1}{V} \sum_{\beta} \Pi^{\beta}_{kn} \frac{\partial \varepsilon^{\beta D}_{uv}}{\partial E_{ij}} A^{\beta} h^{\beta} = \\
& = -\Big[{}^{t}G_{ijuv} - {}^{t}\Big(\frac{\partial f^{0(d)1}_{uv}}{\partial E_{ij}} \Big) \Big] L^{(e)\ell}_{munl} \frac{1}{V} \sum_{\beta} d^{\beta}_{v} \frac{n^{\beta}_{m}}{h^{\beta}} \Pi^{\beta}_{kn} A^{\beta} h^{\beta}
\end{aligned}
\tag{4.7}
$$

with (see (3.17))

$$
\frac{\partial f^{0(d)1}_{uv}}{\partial E_{ij}} = -B'^{-1}_{uvab} L^{(e)\ell}_{bats} \frac{1}{V} \sum_{\beta} \frac{\partial \varepsilon^{\beta D}_{st}}{\partial E_{ij}} A^{\beta} h^{\beta} \tag{4.8}
$$

Considering complex forms (4.7) and (4.8), it appears useful to exploit the consequences of the linearity of the relation $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ supposed for any layer β with open cracks at its boundaries. It implies the linearity of $\mathbf{f}^{0(d)1}$, i.e. the existence of a macroscopic fourth-order tensor $\mathbf{K}'$ such that $\partial f^{0(d)1}_{uv}/\partial E_{ij} = K'_{uvij}$. Note that the "non local" effects of open cracks in the heterogeneous medium represented by $\mathbf{f}^{0(d)1}$ will be described by a linear function of $\mathbf{E}$. Thus, the final form of (4.7) to satisfy is

$$
L^{(e)\ell}_{vunl} \frac{1}{V} \sum_{\beta} \Pi^{\beta}_{kn} \frac{\partial \varepsilon^{\beta D}_{uv}}{\partial E_{ij}} A^{\beta} h^{\beta} = -{}^{t}(G - K')_{ijuv} L^{(e)\ell}_{munl} \frac{1}{V} \sum_{\beta} d^{\beta}_{v} \frac{n^{\beta}_{m}}{h^{\beta}} \Pi^{\beta}_{kn} A^{\beta} h^{\beta} \tag{4.9}
$$

After calculations (see Appendix B for details), one obtains a linear relationship between $\boldsymbol{\varepsilon}^{\beta D}$ and $\mathbf{E}$, i.e. solution to (4.9)

$$
\varepsilon^{\beta D}_{ij} = -Id_{ijmu} d^{\beta}_{v} \frac{n^{\beta}_{m}}{h^{\beta}} (G - K')_{uvkl} E_{lk} + r^{\beta D}_{ij} \tag{4.10}
$$

The constant tensor with respect to $\mathbf{E}$, $\mathbf{r}^{\beta D}$, represents a residual strain induced in the layer β by residual opening of the cracks at its boundaries when $\mathbf{E} = \mathbf{0}$. It is reasonable to think that it is a function of $\mathbf{D}$ and $\overline{\mathbf{D}}$.

In view of (B.5), if (4.10) is satisfied for every βth layer with open cracks at its boundaries, then (4.9) is verified and furthermore $(4.5)_1$ also, but the converse is not true. In other words, from a mathematical viewpoint, the solution is not unique. Nevertheless, the strategy employed in Appendix B, based on the assumption that the relations $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ have the same form for every βth-layer, is in accordance with the Christoffersen framework. Moreover, the result seems pertinent since the "local" strain induced in the layer β

by open cracks at its interfaces depends on damage through $\mathbf{D}$ and $\overline{\mathbf{D}}$ appearing explicitly in $\mathbf{A}'$ and $\mathbf{B}'$, but also on the geometrical features of the layer considered.

It remains now to determine the expression for $\mathbf{K}'$. The latter has to satisfy $\partial f_{uv}^{0(d)1}/\partial E_{ij} = K'_{uvij}$ with $\partial f_{uv}^{0(d)1}/\partial E_{ij}$ given by (4.8) and with $\boldsymbol{\varepsilon}^{\beta D}$ being represented by (4.10). After some manipulations, it follows

$$\mathbf{K}' = [\mathbf{B}' + {}^t(\mathbf{A}' - \mathbf{A})]^{-1} : {}^t(\mathbf{A}' - \mathbf{A}) : \mathbf{G} \tag{4.11}$$

where one has assumed the invertibility of $\mathbf{B}' + {}^t(\mathbf{A}' - \mathbf{A})$ with respect to the identity tensor $\mathbf{Id}^1$. Once relation (4.10) obtained for $\boldsymbol{\varepsilon}^{\beta D}$ is accepted as pertinent, the resulting form (4.11) of $\mathbf{K}'$ is unique. Furthermore, the "non-local" effects of open cracks inside the RVE are now represented by (see (3.17) with (4.10))

$$\mathbf{f}^{0(d)1}(\mathbf{E}, \mathbf{D}, \overline{\mathbf{D}}) = \mathbf{K}' : \mathbf{E} - \mathbf{B}'^{-1} : \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_{\beta} \mathbf{r}^{\beta D} A^{\beta} h^{\beta} \tag{4.12}$$

The second term in (4.12) characterizes the specific contribution of the residual opening of cracks when $\mathbf{E} = \mathbf{0}$. It will be denoted by $\mathbf{f}^{0(\mathrm{d})1\mathrm{Res}}$. One may emphasise the complex structure of $\mathbf{K}'$ involving elastic moduli of both constituents, the tensor $\overline{\mathbf{T}}$ (via $\mathbf{B}'$) related to the material initial morphology and internal organization and $\mathbf{D}$, $\overline{\mathbf{D}}$. This is in perfect accordance with the role of $\mathbf{f}^{0(d)1}$.

Remark: In addition to its linearity, one has supposed the independence of the relation $\boldsymbol{\varepsilon}^{\beta D} = \boldsymbol{\varepsilon}^{\beta D}(\mathbf{E})$ on the set $\{\boldsymbol{\varepsilon}^{fD}\}$. Such an assumption seems reasonable when recalling that $\boldsymbol{\varepsilon}^{\beta D}$ represents the local strain induced in a layer β by open cracks at its own interfaces and that the influence of the distorsion of closed cracks on the strain of this layer is already taken into account through the non local term $\mathbf{f}^{0(d)2}$. Note that without such an assumption, the disconnection in (4.5) would be no longer valid, so that feasibility of obtaining an analytical solution would remain questionable considering the complexity of various couplings involved.

4.3. Macroscopic stress-strain relation

With (4.10)-(4.11), one may formulate the whole elastic model giving local and global responses of the elastic damaged composite in terms of macroscopic variables $\mathbf{E}$, $\{\boldsymbol{\varepsilon}^{fD}\}$ and damage tensors $\mathbf{D}$, $\overline{\mathbf{D}}$. For simplicity, one just reports below the macroscopic stress-strain relation obtained by derivation of the overall free energy given by (4.1)-(4.2) and (A.1) to (A.5), in which (4.10) is substituted for $\boldsymbol{\varepsilon}^{\beta D}$

$$\boldsymbol{\Sigma} = \frac{\partial W}{\partial \mathbf{E}} = \tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}}) : \mathbf{E} + \boldsymbol{\Sigma}^{(d)}(\{\boldsymbol{\varepsilon}^{fD}\}, \mathbf{D}, \overline{\mathbf{D}}) \tag{4.13}$$

$$\begin{aligned}\boldsymbol{\Sigma}^{(d)} &= \underbrace{\mathbf{A} : \mathbf{f}^{0(d)2} + \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_f \boldsymbol{\varepsilon}^{fD} A^f h^f}_{\boldsymbol{\Sigma}^{(d)2}(\{\boldsymbol{\varepsilon}^{fD}\},\mathbf{D},\overline{\mathbf{D}})} + \\ &\quad + \underbrace{\mathbf{A} : \mathbf{f}^{0(d)1Res} + \mathbf{L}^{(e)\ell} : \frac{1}{V}\sum_\beta \mathbf{r}^{\beta D} A^\beta h^\beta}_{\boldsymbol{\Sigma}^{(d)1}(\mathbf{D},\overline{\mathbf{D}})}\end{aligned} \tag{4.14}$$

$$\tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}}) = \langle \mathbf{L}^{(e)} \rangle_V + {}^t(\mathbf{G} - \mathbf{K}') : [\mathbf{H} - {}^t(\mathbf{A}' - \mathbf{A}) - (\mathbf{A}' - \mathbf{A})] : (\mathbf{G} - \mathbf{K}') \tag{4.15}$$

$$H_{ijkl} = L^{(e)\ell}_{mjnl} \overline{D}_{imkn} - B_{ijkl} \tag{4.16}$$

In (4.14), the expression for $\boldsymbol{\Sigma}^{(d)2}$ representing the macroscopic residual stress induced by the blockage of closed cracks remains unchanged, and $\boldsymbol{\Sigma}^{(d)1}$ corresponds now to the residual stress induced by the residual opening of (open) cracks. Direct calculation of $\boldsymbol{\Sigma}$ using (3.13) with $\boldsymbol{\Sigma}^{(v)}$ suppressed, (3.14) and (3.18) in which (4.10) is substituted for $\boldsymbol{\varepsilon}^{\beta D}$, leads to

$$\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma} \rangle_V = [\langle \mathbf{L}^{(e)} \rangle_V - {}^t\mathbf{A}' : (\mathbf{G} - \mathbf{K}')] : \mathbf{E} + \boldsymbol{\Sigma}^{(d)}(\{\boldsymbol{\varepsilon}^{fD}\}, \mathbf{D}, \overline{\mathbf{D}}) \tag{4.17}$$

with $\boldsymbol{\Sigma}^{(d)}$ given by (4.14). With (4.11), one may fortunately prove that $\tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}}) = \langle \mathbf{L}^{(e)} \rangle_V - {}^t\mathbf{A}' : (\mathbf{G} - \mathbf{K}')$ so that (4.17) and (4.13) are equivalent. This equivalence shows that solution (4.10)-(4.11) satisfies explicitly, as expected, (4.4), i.e. $\langle \boldsymbol{\sigma} \rangle_V = \partial \langle w \rangle_V / \partial \mathbf{E}$. Moreover, the degraded elastic moduli tensor $\tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}})$ has now all symmetries (notably the major symmetry) contrarily to $\mathbf{L}(\mathbf{D}, \overline{\mathbf{D}})$ given by (3.14). This result concerning the indicial symmetries of the effective moduli obtained by means of the complementary localization-homogenization procedure stands as the proof for its efficiency. From a theoretical viewpoint, equation (4.15) for the effective moduli seems to be more appropriate than the one in (4.17) since it clearly shows the major symmetry. In the limit cases, where there is no open crack inside the RVE, i.e. for only closed cracks or for the sound material, one may observe that the effective moduli $\tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}})$ correspond, as expected, to those of the sound material: $\mathbf{L} = \langle \mathbf{L}^{(e)} \rangle_V - \mathbf{A} : \mathbf{B}^{-1} : \mathbf{A}$. This result constitutes a new confirmation of the coherence of the complementary localization-homogenization approach.

At last, a comment should be made regarding practical determination of the variables $\{\boldsymbol{\varepsilon}^{fD}\}$ accounting for the frictional locking effect of closed cracks. Considering the hypothesis of no sliding on closed crack lips, a microcrack is, in the present framework, necessarily open before being closed. Moreover, $\boldsymbol{\varepsilon}^{fD}$ for a layer with closed cracks at its boundaries does not evolve as long as the cracks remain closed. Therefore, the components of $\boldsymbol{\varepsilon}^{fD}$ may be calculated from those of $\boldsymbol{\varepsilon}^{\beta D}$ given by (4.10) at the crack closure, precisely when the layer under consideration – initially with open cracks – becomes a layer with closed cracks. The crucial problem is to simultaneously ensure the homogenized energy and stress-response continuity in spite of discontinuity of effective moduli, see e.g. Dragon and Halm (2004). The respective conditions formulated in the context of the scale transition at stake should also give tools to express the tensors $\mathbf{r}^{\beta D}$ in function of $\mathbf{D}$, $\overline{\mathbf{D}}$ and geometrical features of the layer β. Such a strategy is necessarily associated with the formulation of rigorous criteria of unilaterality. This is the aim of present investigations concerning the unilateral effect (i.e. opening/closure transition modelling).

5. Conclusion

A non-classical homogenization method that constitutes an extension of the Christoffersen approach for both viscoelasticity and damage by grain/matrix debonding is presented. The discontinuities have been first introduced in a compatible way with the Christoffersen framework (geometry and kinematics) and following the strategy of this author. It is shown that the direct patterning of the material microstructure and associated local kinematics due to Christoffersen can accommodate the grain/layer discontinuities with just one additional assumption regarding geometry of opposite interfaces of a given layer, introduced in order to make acceptable their simultaneous decohesion (if any) resulting from the hypothesis of the identical displacement gradient $\mathbf{f}^0$ for grains. Moreover, the Christoffersen's morphology and kinematics framework, extended in the presence of damage, offers an advantage to take into account some strain heterogeneity in the matrix phase in estimation of homogenized behaviour (see in (2.16)-(2.18) the dependence of $\mathbf{f}^\alpha$ on morphological features of the layer α and on $\mathbf{f}^{\alpha D}$ if it is debonded).

The solution to the localization-homogenization problem obtained in Section 3 for composites with a viscoelastic matrix, in the presence of interfacial damage, allows one to discern several crucial features. First, the scale transition leads to natural emergence of two macroscopic damage tensors involving granular aspects – a second-order one and a fourth-order one. They describe damage-induced degradation effects and induced anisotropy. These two tensors – in addition to the textural tensor $\overline{\mathbf{T}}$ related to the initial morphology and internal organisation of constituents – allow one to account, in a general 3D context, for coupling the primary anisotropy with the damage-induced one. Local viscoelastic interactions and the macroscopic consecutive long range memory effect are clearly shown to be affected by microcracking. Other remarkable features as recovery of some properties of the sound material under microcrack closure may also be discerned through the comparison with local and global relations obtained for the undamaged material by Nadot-Martin *et al.* (2003). In particular, in the absence of discontinuities, the corresponding expressions for micro- and macro-scale levels reduce to the ones for the sound composite, confirming thus that the methodological coherence is being preserved between both cases (sound and damaged composite) and endorsing specific hypotheses regarding the damaged aggregate and relevant generalization. At last, the advanced scale transition does not make use of the hypothesis of non-interacting cracks (each microcrack is not considered as isolated in an infinite medium) so that some "non-local" damage effects may be identified at both scales. One should realize in the same time that this does not mean that defects interact in the sense pointed out by e.g. Kachanov (1994). Indeed, on can note in particular that the influence of "non-local" damage effects within the RVE, embodied by the term $\mathbf{f}^{0(\mathrm{d})}$, on the strain of any layer is just pondered by morphological features of the layer considered, and does not involve any distance separating this layer from "remote" defects.

Some superfluous damage-induced strain-like variables related to open cracks (i.e. $\{\boldsymbol{\varepsilon}^{\beta D}\}$) are still explicitly present at this stage. Their status as well as some other properties of homogenized expressions indicate that further analysis is needed to obtain a net and thermodynamically consistent formulation. The latter has been achieved via complementary localization-homogenization analysis under notable simplification regarding behaviour of constituents: only elastic-damaged system has been considered in Section 4. The local strain $\boldsymbol{\varepsilon}^{\beta D}$ induced in any layer β with open cracks at its boundaries is thus expressed as a function of the macroscopic variable $\mathbf{E}$, damage tensors $\mathbf{D}$, $\overline{\mathbf{D}}$ and geometrical features of the layer at stake in such a way that the homogenized stress derives explicitly from the global free energy. By giving access to the effective moduli in a direct and thermodynamically consistent manner, such a preliminary analysis performed in the elastic context will stand as a reference for further viscoelasticity-damage complementary localization-homogenization approach. The latter will also include replacement of the set of relaxation internal variables $\{\boldsymbol{\gamma}^{\alpha}\}$ by a single variable as it was done in Nadot-Martin *et al.* (2003) for the sound material.

Further work will include – apart from fully viscoelasticity-damage complementary study – a detailed treatment of unilateral phenomena and damage evolution. It is to be noted that the strategy regarding modelling of the unilateral effect proposed in the elastic context at the end of Section 4, remains valid for viscoelastic constituents.

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A. Appendix

This appendix presents detailed expressions of the terms W^i for $i = 2, \ldots, 6$ figuring in homogenized free energy (4.1) obtained for the damaged elastic aggregate in Subsection 4.1.

$$\begin{aligned} W^2(\mathbf{E}, \{\boldsymbol{\varepsilon}^{\beta D}\}, \mathbf{D}, \overline{\mathbf{D}}) = E_{uv}\Big[(A - {}^tG : B - B')_{vukl} f^{0(d)1}_{lk} + \\ - {}^tG_{vurm} L^{(e)\ell}_{srkl} \frac{1}{V} \sum_{\beta} \mp^{\beta}_{ms} \varepsilon^{\beta D}_{lk} A^{\beta} h^{\beta}\Big] \end{aligned} \tag{A.1}$$

$$\begin{aligned} W^3(\mathbf{E}, \{\boldsymbol{\varepsilon}^{fD}\}, \mathbf{D}, \overline{\mathbf{D}}) = \\ = E_{uv}\Big\{[A - {}^tG : (B - B')]_{vukl} f^{0(d)2}_{lk} + L^{(e)\ell}_{vukl} \frac{1}{V} \sum_{f} \varepsilon^{fD}_{lk} A^f h^f\Big\} \end{aligned} \tag{A.2}$$

$$\begin{aligned} W^4(\{\boldsymbol{\varepsilon}^{\beta D}\}, \mathbf{D}, \overline{\mathbf{D}}) = \Big(\frac{1}{2} f^{0(d)1}_{uv} B_{vukl} + L^{(e)\ell}_{mlvu} \frac{1}{V} \sum_{\beta} \Pi^{\beta}_{km} \varepsilon^{\beta D}_{uv} A^{\beta} h^{\beta}\Big) f^{0(d)1}_{lk} + \\ + L^{(e)\ell}_{vukl} \frac{1}{2V} \sum_{\beta} \varepsilon^{\beta D}_{uv} \varepsilon^{\beta D}_{lk} A^{\beta} h^{\beta} \end{aligned} \tag{A.3}$$

$$\begin{aligned} W^5(\{\boldsymbol{\varepsilon}^{\beta D}\}, \{\boldsymbol{\varepsilon}^{fD}\}, \mathbf{D}, \overline{\mathbf{D}}) = \\ = \Big[f^{0(d)1}_{uv} (B - B')_{vukl} + L^{(e)\ell}_{mlvu} \frac{1}{V} \sum_{\beta} \Pi^{\beta}_{km} \varepsilon^{\beta D}_{uv} A^{\beta} h^{\beta}\Big] f^{0(d)2}_{lk} \end{aligned} \tag{A.4}$$

$$\begin{aligned} W^6(\{\boldsymbol{\varepsilon}^{fD}\}, \mathbf{D}, \overline{\mathbf{D}}) = \\ = f^{0(d)2}_{uv} \Big(\frac{1}{2} B - B'\Big)_{vukl} f^{0(d)2}_{lk} + L^{(e)\ell}_{vukl} \frac{1}{2V} \sum_{f} \varepsilon^{fD}_{uv} \varepsilon^{fD}_{lk} A^f h^f \end{aligned} \tag{A.5}$$

B. Appendix

This appendix deals with the determination of a linear relation between $\boldsymbol{\varepsilon}^{\beta D}$ and $\mathbf{E}$ that satisfies differential equation (4.9) established in Subsection 4.2. Since there is only one equation for M unknown functions, where M denotes the number of layers with open cracks at their boundaries, it is mathematically impossible to find these functions in a unique manner. One proposes here a reasonable way based on the assumption that the above mentioned relations have the same structure for every βth layer.

Equation (4.9) is satisfied if

$$L^{(e)\ell}_{vunl}\Pi^{\beta}_{kn}\frac{\partial\varepsilon^{\beta D}_{uv}}{\partial E_{ij}} = -{}^{t}(G-K')_{ijuv}L^{(e)\ell}_{munl}d^{\beta}_{v}\frac{n^{\beta}_{m}}{h^{\beta}}\Pi^{\beta}_{kn} \tag{B.1}$$

for every βth layer with open cracks at their boundaries. Consider a single βth layer. This particular layer verifies (B.1) if

$$L^{(e)\ell}_{vukl}\frac{\partial\varepsilon^{\beta D}_{uv}}{\partial E_{ij}} = -{}^{t}(G-K')_{ijuv}L^{(e)\ell}_{mukl}d^{\beta}_{v}\frac{n^{\beta}_{m}}{h^{\beta}} \tag{B.2}$$

Using the invertibility of $\mathbf{L}^{(e)\ell}$, (B.2) becomes equivalent to

$$\frac{\partial\varepsilon^{\beta D}_{rs}}{\partial E_{ij}} = -Id_{rsmu}d^{\beta}_{v}\frac{n^{\beta}_{m}}{h^{\beta}}(G-K')_{uvij} \tag{B.3}$$

Finally, one obtains $\boldsymbol{\varepsilon}^{\beta D}$ in terms of $\mathbf{E}$ by solving (B.3)

$$\varepsilon^{\beta D}_{ij} = -Id_{ijmu}d^{\beta}_{v}\frac{n^{\beta}_{m}}{h^{\beta}}(G-K')_{uvkl}E_{lk} + r^{\beta D}_{ij} \tag{B.4}$$

where $\mathbf{r}^{\beta D}$ is a constant tensor with respect to $\mathbf{E}$. This simple calculation provides a linear form for $\boldsymbol{\varepsilon}^{\beta D}$ in function of $\mathbf{E}$ that, when satisfied for every layer β with open cracks at their boundaries, leads to (4.9)

$$\text{(B.4)}\ \forall\beta \Leftrightarrow\ \text{(B.2)}\ \forall\beta \Rightarrow\ \text{(B.1)}\ \forall\beta \Rightarrow\ \text{(4.9)} \tag{B.5}$$

Significant symbols

Morphological parameters and tensors

h^{α} – thickness of αth layer
$\boldsymbol{n}^{\alpha}$ – unit normal vector defining orientation of αth layer

$\boldsymbol{d}^{\alpha}$	–	vector linking centroids of grains interconnected by αth layer
A^{α}	–	projected area of αth layer
$\boldsymbol{c}^{\alpha}$	–	vector connecting centres of two opposite boundaries of αth layer
c	–	ratio of layer volume to volume V of grains and layers
$\boldsymbol{\Pi}^{\alpha}$	–	second-order tensor accounting for geometry of αth layer
$\overline{\mathbf{T}}$	–	fourth-order structural tensor accounting for morphology and internal organization of constituents inside the Representative Volume Element (RVE)

Kinematical quantities

$\mathbf{F}$	–	global (macroscopic) displacement gradient
$\mathbf{f}^{0}$	–	displacement gradient of grains
$\mathbf{f}^{0(r)}$	–	reversible part of $\mathbf{f}^{0}$
$\mathbf{f}^{0(v)}$	–	viscous part of $\mathbf{f}^{0}$ accounting for viscoelastic interactions between constituents
$\mathbf{f}^{0(d)}$	–	damage-induced part of $\mathbf{f}^{0}$ accounting for "non local" effects of whole set of defects inside RVE
$\mathbf{f}^{0(d)1}$	–	"non local" effects of open defects inside RVE
$\mathbf{f}^{0(d)2}$	–	"non local" effects of closed defects inside RVE
$\mathbf{f}^{\alpha}$	–	displacement gradient of αth layer
$\mathbf{f}^{\alpha D}$	–	contribution of defects located at boundaries of a debonded layer α to its displacement gradient $\mathbf{f}^{\alpha}$
$\boldsymbol{u}^{GA}, \boldsymbol{u}^{GB}$	–	displacement field of grains GA and GB, respectively
$\boldsymbol{u}^{\alpha}$	–	displacement field of αth layer
$\boldsymbol{b}^{\alpha 1}, \boldsymbol{b}^{\alpha 2}$	–	displacement discontinuity vectors across interfaces I_1^{α} and I_2^{α} of a debonded layer α

Strain-like quantities

$\mathbf{E}$	–	global (macroscopic) strain tensor
γ^{α}	–	viscoelastic internal second-order tensorial variable accounting for relaxation state of αth layer
$\boldsymbol{\varepsilon}$	–	local strain tensor field
$\boldsymbol{\varepsilon}^{(v)}, \boldsymbol{\varepsilon}^{(d)}$	–	respectively viscous and damage-induced parts of local strain field
$\boldsymbol{\varepsilon}^{0(v)}, \boldsymbol{\varepsilon}^{0(d)}$	–	respectively viscous and damage-induced parts of strain tensor for grains
$\boldsymbol{\varepsilon}^{\alpha(v)}, \boldsymbol{\varepsilon}^{\alpha(d)}$	–	respectively viscous and damage-induced parts of strain tensor for αth layer $\boldsymbol{\varepsilon}^{\alpha} = \mathrm{Sym}\,\mathbf{f}^{\alpha}$

$\boldsymbol{\varepsilon}^{\alpha D} = \mathrm{Sym}\,\mathbf{f}^{\alpha D}$	–	for a debonded layer α, "local" contribution of its own defects to its strain $\boldsymbol{\varepsilon}^{\alpha} = \mathrm{Sym}\,\mathbf{f}^{\alpha}$
$\boldsymbol{\varepsilon}^{\beta D} = \mathrm{Sym}\,\mathbf{f}^{\beta D}$	–	for a layer β with open defects at its boundaries, "local" contribution of its own defects to its strain $\boldsymbol{\varepsilon}^{\beta} = \mathrm{Sym}\,\mathbf{f}^{\beta}$
$\boldsymbol{\varepsilon}^{f D} = \mathrm{Sym}\,\mathbf{f}^{f D}$	–	for a layer f with closed defects at its boundaries, "local" contribution of its own defects to its strain $\boldsymbol{\varepsilon}^{f} = \mathrm{Sym}\,\mathbf{f}^{f}$. Internal variable accounting for distortion due to blockage of corresponding closed defects
$\mathbf{r}^{\beta D}$	–	residual strain induced in a layer β by residual opening of defects at its boundaries when $\mathbf{E} = \mathbf{0}$

Stresses

$\boldsymbol{\Sigma}$	–	global (homogenized) stress tensor
$\boldsymbol{\Sigma}^{(v)}, \boldsymbol{\Sigma}^{(d)}$	–	respectively viscous and damage-induced parts of homogenized stress tensor $\boldsymbol{\Sigma}$
$\boldsymbol{\Sigma}^{(d)1}$	–	contribution of open defects to the damage-induced stress tensor $\boldsymbol{\Sigma}^{(d)}$
$\boldsymbol{\Sigma}^{(d)2}$	–	contribution of closed defects to the damage-induced stress tensor $\boldsymbol{\Sigma}^{(d)}$, macroscopic residual stress tensor corresponding to blockage of closed defects inside RVE
$\boldsymbol{\sigma}^{0}, \boldsymbol{\sigma}^{\alpha}$	–	average stress tensors in grains and in αth layer, respectively
$\boldsymbol{t}^{\alpha}$	–	total force transmitted through interfacial αth layer

Local and global moduli and essential tensors involved

$\mathbf{L}^{(e)\ell}, \mathbf{L}^{(v)}$	–	fourth-order tensors of elastic and viscous moduli for matrix
$\mathbf{L}^{0}$	–	fourth-order tensor of elastic moduli for grains
$\mathbf{C}$	–	elastic concentration tensor field
$\mathbf{C}^{0}, \mathbf{C}^{\alpha}$	–	elastic concentration tensor for grains and αth layer, respectively
$\mathbf{L}$	–	reversible global moduli tensor for sound material (without damage)
$\mathbf{L}(\mathbf{D}, \overline{\mathbf{D}})$	–	"incomplete" reversible global moduli tensor in presence of damage
$\tilde{\mathbf{L}}(\mathbf{D}, \overline{\mathbf{D}})$	–	reversible global moduli tensor (after complementary analysis)
$\mathbf{A}, \mathbf{B}$	–	fourth-order tensors involved in local and global response expressions

$\mathbf{A}', \mathbf{B}'$	–	damage degraded forms (via $\mathbf{D}$ and $\overline{\mathbf{D}}$) of the tensors $\mathbf{A}$ and $\mathbf{B}$
$\mathbf{D}, \overline{\mathbf{D}}$	–	second-order and fourth-order damage tensors

Identity tensors and particular operators

$\boldsymbol{\delta}$	–	symmetric second-order identity tensor
$\mathbf{Id}$	–	classical fourth-order identity tensor defined by $Id_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$
$\mathbf{Id}^1$	–	fourth-order identity tensor defined by $Id^1_{ijkl} = \delta_{il}\delta_{jk}$
$\langle\cdot\rangle_V$	–	volume average
:	–	tensorial double contraction defined by: $C_{ijkl} = A_{ijmn}B_{nmkl}$ if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are fourth-order tensors, $C_{ij} = A_{ijkl}B_{lk}$ if $\mathbf{A}$ is fourth-order tensor and $\mathbf{B}$ second-order one

References

1. Andrieux S., Bamberger Y., Marigo J.J., 1986, Un modèle de matériau microfissuré pour les bétons et les roches, *J. Méca. Théor. Appl.*, **5**, 3 471-513

2. Basista M., Gross D., 1997, Internal variable representation of microcrack induced inelasticity in brittle materials, *Int. J. Damage Mechanics*, **6**, 300-316

3. Beurthey S., Zaoui A., 2000, Structural morphology and relaxation spectra of viscoelastic heterogeneous materials, *Eur. J. Mech. A/Solids*, **19**, 1-16

4. Brenner R., Masson R., Castelnau O., Zaoui A., 2002, A "quasi-elastic" affine formulation for the homogenized behaviour of nonlinear viscoelastic polycrystals and composites, *Eur. J. Mech. A/Solids*, **21**, 943-960

5. Christoffersen J., 1983, Bonded granulates, *J. Mech. Phys. Solids*, **31**, 55-83

6. Dragon A., Halm D., 2004, *Damage Mechanics. Some Modelling Challenges*, Centre of Excellence AMAS, Warsaw

7. Kachanov M., 1994, Elastic solids with many cracks and related problems, *Advances in Applied Mechanics*, **30**, 259-445

8. Moulinec H., Suquet P., 2003, Intraphase strain heterogeneity in non linear composites: a computational approach, *Eur. J. Mech A/Solids*, **22**, 751-770

9. Nadot-Martin C., Trumel H., Dragon A., 2003, Morphology-based homogenization method for viscoelastic particulate composites, Part I: Viscoelasticity sole, *Eur. J. Mech. A/Solids*, **22**, 89-106

10. PONTE CASTAÑEDA P., 2002, Second-order homogenization estimates for non linear composites incorporating field fluctuations: I – Theory., *J. Mech. Phys. Solids*, **50**, 737-757

Modelowanie uszkodzenia w granulowanych kompozytach lepkosprężystych przy pomocy podejścia wieloskalowego

Streszczenie

Celem tej publikacji jest sformułowanie wieloskalowego modelu mikromechanicznego dla granulowanych kompozytów o wysokim stopniu upakowania inkluzji w osnowie lepkosprężystej. Przedstawiony model, będący rozwinięciem morfologicznego podejścia Christoffersena (1983) i Nadot-Martin i in. (2003) w zakresie małych odkształceń lepkosprężystych, polega na wprowadzeniu do analizy dodatkowego mechanizmu uszkodzenia – mikropękania na granicy inkluzji i osnowy. Mikroszczeliny na granicy inkluzji i osnowy uwzględniono w hipotezie geometrycznej i kinematycznej metody Christoffersena. Następnie, wyznaczono lokalne oraz uśrednione pola naprężenia dla zadanego stanu uszkodzenia (tzn. dla zadanej liczby otwartych i zamkniętych mikroszczelin przy pominięciu poślizgów na powierzchniach mikroszczelin zamkniętych). Porównanie z wynikami uzyskanymi przez Nadot-Martin i in. (2003) dla nieuszkodzonego kompozytu lepkosprężystego pozwoliło na określenie wpływu uszkodzenia na poziomie lokalnym i globalnym. Na koniec, podstawowy model wieloskalowy uzupełniono o drugą część sformułowania, która pozwoliła usunąć pewne nadmiarowe odkształcenia związane z mikroszczelinami otwartymi, czyniąc cały model termodynamicznie spójnym. Ta druga część modelu wieloskalowego jest przeprowadzona przy założeniu upraszczającym, polegającym na (tymczasowym) pominięciu efektów lepkosprężystych.

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