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# CHAOTIC VIBRATION OF AN AUTOPARAMETRICAL SYSTEM WITH A NON IDEAL SOURCE OF POWER

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This paper studies the dynamical coupling between energy sources and the response of a two degrees of freedom autoparametrical system, when the excitation comes from an electric motor (with unbalanced mass $m_0$), which works with limited power supply. The investigated system consists of a pendulum of the length $l$ and mass $m$, and a body of mass $M$ suspended on a flexible element. In this case, the excitation has to be expressed by an equation describing how the energy source supplies the energy to the system. The non-ideal source of power adds one degree of freedom, which makes the system have three degrees of freedom. The system has been searched for known characteristics of the energy source (DC motor). The equations of motion have been solved numerically. The influence of motor speed on the phenomenon of energy transfer has been studied. Near the internal and external resonance region, except for different kinds of periodic vibration, chaotic vibration has been observed. For characterizing an irregular chaotic response, bifurcation diagrams and time histories, power spectral densities, Poincaré maps and maximal exponents of Lyapunov have been developed.

*Key words:* nonlinear dynamics, non ideal system, energy transfer, chaos

## 1. Introduction

Depending on whether excitation is influenced or not by the response of a system, the vibrating system may be called ideal or non-ideal. When the forcing is independent of the system it acts on, then it is called ideal. The ideal problems are the traditional ones. Formally, the excitation is expressed as a pure function of time, for example by a sinusoidal excitation. In this case, the excitation is independent of the system response. On the other hand, when the forcing function depends on the response of the system, it is said to be non-ideal. When we use a non-ideal source of power instead of an ideal one, the excitation should be presented as a function which depends on the response of the system. In this case, the non-ideal source of power can not be expressed as a pure function of time but as an equation that relates the source of energy to the system. Then the effect of energy supply is described by another different equation, increasing the number of degrees of freedom. A non-ideal source of power is for example a DC motor with an unbalanced mass.

The first detailed study on non-ideal vibrating systems is a monograph by Kononienko (1969). He obtained satisfactory results through the comparison of numerical analysis and approximated methods. According to Kononienko (1969), characteristics of an oscillatory system become dependent on the properties of the energy source. After that publication, the problem of non-ideal vibrating systems has been investigated by a number of authors. Simulations of similar models were described by Giergiel (1990). The non-ideal problems were presented by Evan-Ivanowski (1976) or Nayfeh and Mook (1979). These authors showed that, sometimes, the dynamical coupling between energy sources and the structural response must not be ignored in real engineering problems. A complete review on different theories on non-ideal vibrating systems were discussed and presented by Balthazar *et al.* (2003). Non-ideal models were researched by Krasnopolskaja and Shvets (1987). In Belato *et al.* (1999), the authors studied a non-ideal similar system consisting of a pendulum whose support point vibrated along a horizontal guide by a two-bar linkage driven from a limited power DC motor. Vibrations of ideal and non-ideal parametrical and self-excited models were described by Půst (1995) and Warmiński *et al.* (2001). Calvalca *et al.* (1999) studied a non-linear model for the Laval rotor with an unlimited power source. A model for flexible slewing structures with DC motors was investigated by Fenili *et al.* (2003). Possibilities of the existence of regular and irregular motion in non-ideal parametrical models were presented by Warmiński (2001), Belato *et al.* (2001) or Tsuchida *et al.* (2003). Sado and Kot (2002, 2003) investigated a non-ideal autoparametrical system, where the influence of linear damping on energy transfer between modes of vibration was studied.

This paper illustrates results of numerical simulation of a non–ideal autoparametrical system with non-linear damping put on the main mass $M$ and on the pendulum. The present work shows that in this type of a non-ideal system, one mode of vibrations may excite or damp another mode, and near the resonance regions, except for multiperiodic and quasiperiodic vibrations, chaotic motion may appear as well. To prove the chaotic character of this vibration, bifurcation diagrams for different damping parameters, time histories, power spectral densities (using FFT), Poincaré maps and Lyapunov exponents are developed. These descriptors are devoted to observe chaos, and to better understand it (Moon, 1987; Baker and Gollub, 1996). Due to nonlinearities and a coupled nature of the equations of motion, numerical solutions are used.

## 2. A model of an autoparametrical system with a non ideal source of power

The investigated model of an autoparametrical two-degrees-of-freedom system with a non-ideal source of power is shown in Fig. 1. The system consists of a pendulum and a body of mass $M$ suspended on a flexible element characterized by a linear elasticity $k$ and a non-linear viscous damping. The pendulum with a weightless rod of the length $l$ and a lumped mass $m$ is mounted to the body $M$. It is assumed that the non-linear viscous damping force applied to the hinge opposes motion of the pendulum. The body of mass $M$ is subjected to an excitation by an electric motor with an unbalanced mass $m_0$. In this case, this DC motor is the non-ideal source of power.

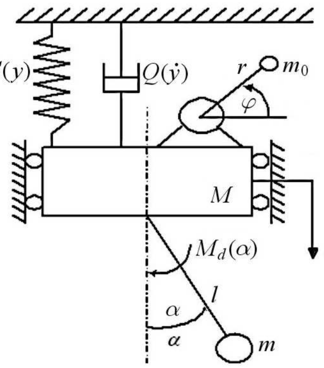


Fig. 1. A non-ideal model of an autoparametrical system

The non-ideal source of power adds one degree of freedom, thus the system has three degrees of freedom. The generalized co-ordinates are: the vertical displacement $y$ of the main mass $M$, the angular displacement $\alpha$ of the pendulum measured from the vertical line, and the co-ordinate $\varphi$ which describes the angular displacement of the unbalanced mass $m_0$ measured from the horizontal line.

It is assumed that the elasticity force is $S(y) = k(y+y_{st})$, where $y_{st}$ is the static vertical displacement which can be found from the rel;ation: $(M+m+m_0)g = ky_{st}$, where $g$ is the acceleration of gravity and $k$ is the coefficient of elasticity. Also is assumed that the damping force acting on the body $M$ is $Q_1(\dot{y}) = C_1\dot{y} + C_3\dot{y}^3$, while the resistant moment acting on the pendulum is $M_d(\dot{\alpha}) = C_2\dot{\alpha} + C_4\dot{\alpha}^3$, where $C_1$, $C_2$, $C_3$, $C_4$ are constant coefficients of damping.

Vibrations have been researched around the static point of balance and the equations of motion have been derived from Lagranges formula. The kinetic $(T)$ and potential energy $(V)$ of the system are

$$\begin{aligned} T &= \frac{1}{2}(I_0 + m_0r^2)\dot{\varphi}^2 + \frac{1}{2}(M + m + m_0)\dot{y}^2 + \frac{1}{2}ml^2\dot{\alpha}^2 + \\ &\quad -m_0r\dot{y}\dot{\varphi}\cos\varphi - ml\dot{y}\dot{\alpha}\sin\alpha \\ V &= \frac{1}{2}k(y + y_{st})^2 + m_0g(r\sin\varphi - y) + mg(l - l\cos\alpha - y) - Mgy \end{aligned} \tag{2.1}$$

The equations of motion of the system take the following form

$$\begin{aligned} &(M + m + m_0)\ddot{y} - ml\dot{\alpha}^2\cos\alpha - ml\ddot{\alpha}\sin\varphi + m_0r\dot{\varphi}^2\sin\varphi + \\ &\qquad -m_0r\ddot{\varphi}\cos\varphi + ky + C_1\dot{y} + C_3\dot{y}^3 = 0 \\ &ml^2\ddot{\alpha} + C_2\dot{\alpha} + C_4\dot{\alpha}^3 - m\ddot{y}l\sin\alpha + mgl\sin\alpha = 0 \\ &(I + m_0r^2)\ddot{\varphi} - m_0\ddot{y}r\cos\varphi + m_0gr\cos\varphi = L(\dot{\varphi}) - H(\dot{\varphi}) \end{aligned} \tag{2.2}$$

where $L(\dot{\varphi})$ is the driving torque of the DC motor and $H(\dot{\varphi})$ is the resistance torque.

The following dimensionless definitions have been introduced in Eqs (2.2)

$$\begin{array}{lll} \tau = \omega_1 t & y_1 = \dfrac{y}{y_{st}} & \omega_1^2 = \dfrac{k}{M + m + m_0} \\ \omega_2^2 = \dfrac{g}{l} & \beta = \dfrac{\omega_1}{\omega_2} & a_1 = \dfrac{m_0r}{(M + m + m_0)y_{st}} \\ a_2 = \dfrac{m}{M + m + m_0} & \gamma_2 = \dfrac{C_2}{ml^2\omega_1} & \gamma_1 = \dfrac{C_1}{(M + m + m_0)\omega_1} \\ \gamma_3 = \dfrac{C_3l^2\omega_1}{M + m + m_0} & \gamma_4 = \dfrac{C_4\omega_1}{ml^2} & q = \dfrac{m_0ry_{st}}{I_0 + m_0r^2} \\ G(\dot{\varphi}) = L(\dot{\varphi}) - H(\dot{\varphi}) & G_1(\dot{\varphi}) = \dfrac{G(\dot{\varphi})}{(I_0 + m_0r^2)\omega_1^2} & \end{array} \tag{2.3}$$

The characteristic curves $G_1(\dot{\varphi})$ of the energy source (DC motor) are assumed to be straight lines: $G_1(\dot{\varphi}) = u_1 - u_2\dot{\varphi}$, where the parameter $u_1$ is related to the voltage, and $u_2$ is a constant parameter for each model of the motor considered. The voltage is the control parameter of the problem.

After transformations, the equations of motion can be written in the form

$$
\begin{aligned}
\ddot{\alpha} &= [(a_1\beta^2\dot{\varphi}^2\sin\varphi - a_2\dot{\alpha}^2\cos\alpha + \beta^2 y_1 + \beta^2\gamma_1\dot{y}_1 + \beta^6\gamma_3\dot{y}_1^3)\sin\alpha + \\
&\quad -(qa_1\cos^2\varphi - 1)(\beta^2\sin\alpha + \gamma_2\dot{\alpha} + \gamma_4\dot{\alpha}^3) + \\
&\quad -a_1\beta^2\cos\varphi\cos\alpha(u_1 - u_2\dot{\varphi} - q\cos\varphi)]\frac{1}{a_2\sin^2\alpha + a_1 q\cos^2\varphi - 1} \\
\ddot{\varphi} &= \Big[\Big(a_1\dot{\varphi}^2\sin\varphi - a_2\frac{1}{\beta^2}\dot{\alpha}^2\cos\alpha + y_1 + \gamma_1\dot{y}_1 + \beta^4\gamma_3\dot{y}_1^3\Big)q\cos\varphi + \\
&\quad +a_2 q\cos\varphi\sin\alpha\Big(\sin\alpha + \frac{1}{\beta^2}\gamma_2\dot{\alpha} + \frac{1}{\beta^2}\gamma_4\dot{\alpha}^3\Big) + \\
&\quad +(u_1 - u_2\dot{\varphi} - q\cos\varphi)(a_2\sin^2\alpha - 1)\Big]\frac{1}{a_2\sin^2\alpha + a_1 q\cos^2\varphi - 1} \\
\ddot{y}_1 &= \Big[a_1\dot{\varphi}^2\sin\varphi - a_2\frac{1}{\beta^2}\dot{\alpha}^2\cos\alpha + y_1 + \gamma_1\dot{y}_1 + \beta^4\gamma_3\dot{y}_1^3 + \\
&\quad +\Big(sin\alpha + \frac{1}{\beta^2}\gamma_2\dot{\alpha} + \frac{1}{\beta^2}\gamma_4\dot{\alpha}^3\Big)a_2\sin\alpha + \\
&\quad -(u_1 - u_2\dot{\varphi} - q\cos\varphi)a_1\cos\varphi\Big]\frac{1}{a_2\sin^2\alpha + a_1 q\cos^2\varphi - 1}
\end{aligned}
\tag{2.4}
$$

## 3. Numerical simulation results

The equations of motion have been solved numerically by the Runge-Kutta method with a variable step length. Calculations have been done for different values of the system parameters and for the following parameters of the engine: $u_1 = 0.2$ to $u_1 = 4$, $u_2 = 1.5$ (for DC motor), where $u_1$ is the control parameter which depends on voltage and the parameter $u_2$ which depends on the type of energy source. The calculations incorporated the following initial conditions: $\dot{\varphi}(0) = 1,\ \alpha(0) = 0.005^\circ,\ \varphi(0) = y(0) = \dot{y}(0) = \dot{\alpha}(0) = 0$ and the parameters $\beta = 0.5,\ q = 0.2$.

The resonant curves for the body of mass $M$ and for the pendulum with damping put on the mass $M$ for the conditions of the autoparametric main internal resonance are shown in Fig. 2. There are three peak amplitudes, resulting from strong interactions between the mass $M$ and the pendulum. The first peak is for the control parameter $u_1 = 0.78$, the second one for $u_1 = 1.465$ and the third for $u_1 = 1.565$.

Near the internal and external resonances, depending on the selection of parameters of a physical system, the amplitudes of vibrations of the coupled system may be related differently. The system presents some interesting nonlinear phenomena. Motions $y_1$ and $\alpha$ are periodic, multi-periodic or quasi-periodic,

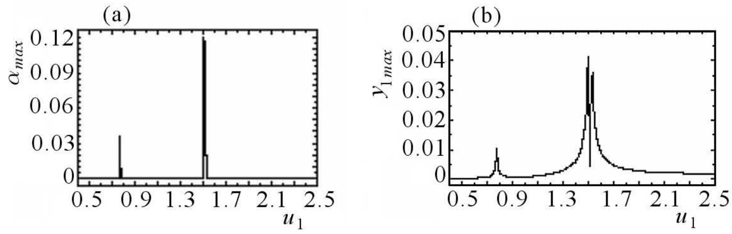


Fig. 2. Amplitudes $\alpha_{max}$ (a) and $y_{1max}$ (b) versus the control parameter $u_1$ for: $u_2 = 1.5$, $a_1 = 0.001$, $a_2 = 0.1$, $q = 0.2$, $\beta = 0.5$, $\gamma_1 = 0.01$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

but sometimes motions of the mass $M$ and the pendulum are chaotic. For characterizing irregular chaotic response forms ang transition zones between one and another type of regular steady resonant motion, bifurcation diagrams are developed. These phenomena can be more easily observed in terms of displacement, sometimes velocities, so diagrams are presented for both. Exemplary results for small damping put on the mass $M$ near the internal resonance (near the principal autoparametric resonance for $\beta = 0.5$) versus the control parameter $u_1$ for displacements $y_1$ and $\alpha$ are shown in Fig. 3, and for velocities $d\varphi/d\tau$ in Fig. 4.

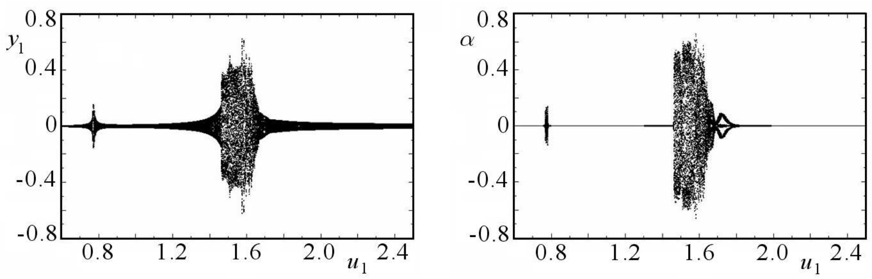


Fig. 3. Bifurcation diagrams for $y_1$ and $\alpha$ for: $u_2 = 1.5$, $q = 0.2$, $\beta = 0.5$, $a_1 = 0.01$, $a_2 = 0.3$, $\gamma_1 = 0.001$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

As we can see in diagrams presented in Fig. 3, motion of the mass $M$ and the pendulum have different characters: may be periodic, multi-periodic, quasi-periodic or irregular. Next, segments of the bifurcation diagrams in a tensioned scale are presented. In Fig. 4 diagrams corresponding to $y_1$, $\alpha$ and $d\varphi/d\tau$ for $u_1 \in (1.49, 1.51)$ are given. As can be seen in Fig. 4, velocity of DC motor has different values.

As it can be seen from these bifurcation diagrams, several phenomena can be observed: the existence of simple or chaotic atractors, and various bifurcations. All these phenomena have to be verified in the phase space. So next the time histories, power spectral densities (their fast Fourier transforms – FFT), Poincaré maps and the largest Lyapunov exponents corresponding to

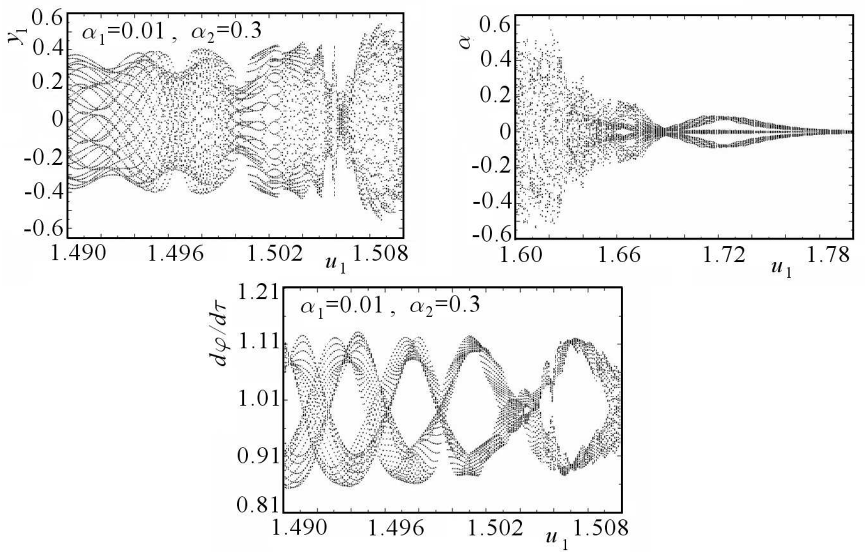


Fig. 4. Bifurcation diagrams for $y_1$, $\alpha$ and $d\varphi/d\tau$ in the region $u_1 \in (1.49, 1.51)$ for: $u_2 = 1.5$, $q = 0.2$, $\beta = 0.5$, $a_1 = 0.01$, $a_2 = 0.3$, $\gamma_1 = 0.001$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

the coordinates $y_1$ and $\alpha$ have been determined. Exemplary results for the control parameter $u_1 = 1.499$ are presented in Fig. 5, for $u_1 = 1.504$ in Fig. 6 and for $u_1 = 1.72$ in Fig. 7.

As can be seen from these diagrams, the responses for presented values of the control parameter $u_1$ are chaotic (the motions look like irregular, the frequency spectra are continuous, the Poincaré's maps trace strange atractors and the largest Lyapunov exponents corresponding to the coordinates $y_1$ and $\alpha$ are positive. As can be seen in Fig. 7, in this non-ideal system after a long time a jump in the amplitudes is sometimes possible, so we should investigate these problems in a larger period of time.

## 4. Conclusions

This work is concerned with the problem of nonlinear dynamical motion of a non-ideal vibrating system with autoparametric coupling. Several interesting phenomena have been presented. The influence of linear and non-linear damping parameters on the energy transfer cycle has been observed. The behaviour of the system near the internal and external resonance frequencies is very important. Depending on the selection of physical system parameters, the amplitudes of vibrations of coupled bodies may be related differently. It has been shown that the examined system exhibits very rich non-linear dyna-

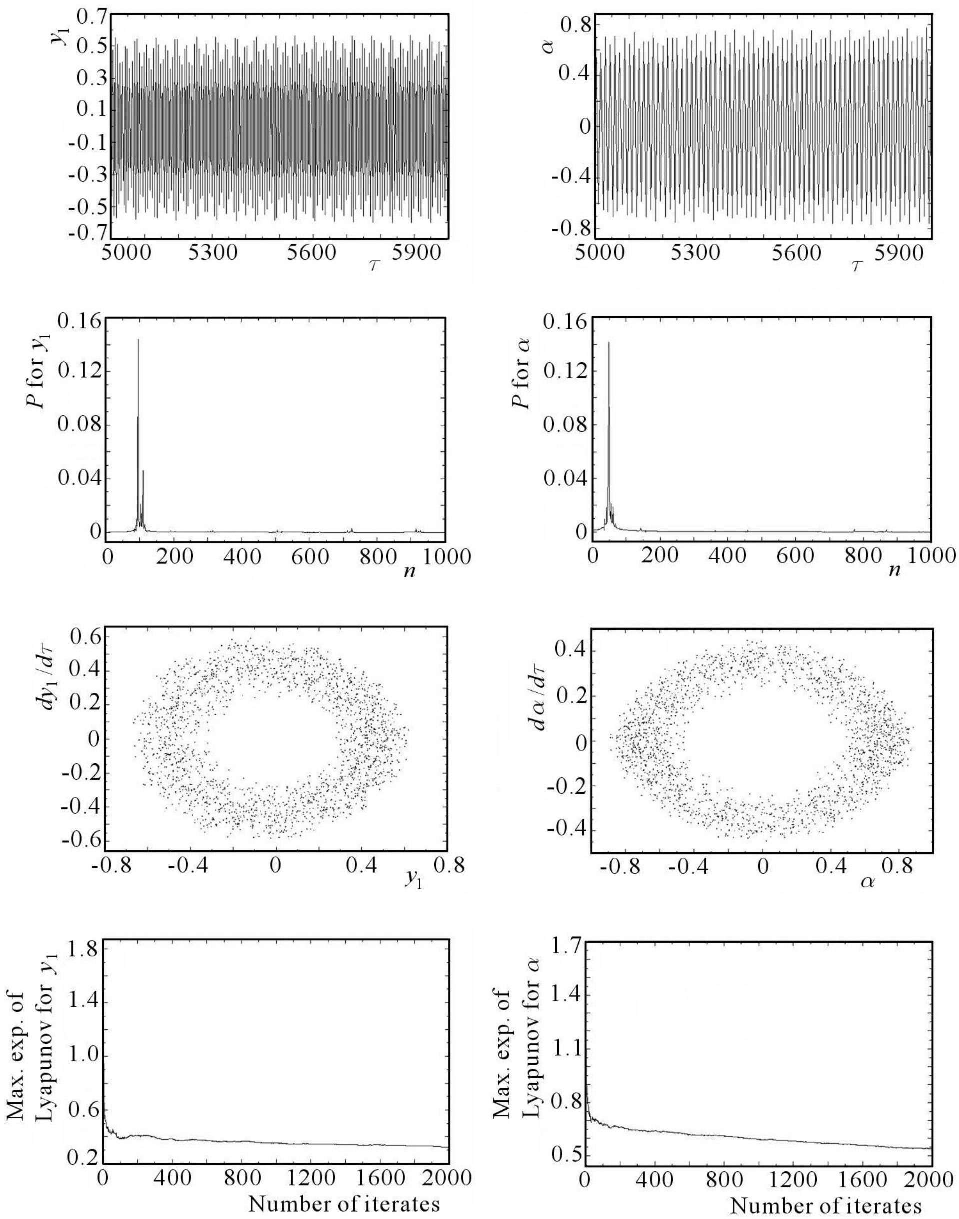


Fig. 5. Time histories, power spectral densities, Poincaré's maps and maximum Lyapunov exponents corresponding to coordinates $y_1$ and $\alpha$ for the control parameter $u_1 = 1.499$ and for: $u_2 = 1.5$, $q = 0.2$, $\beta = 0.5$, $a_1 = 0.01$, $a_2 = 0.3$, $\gamma_1 = 0.001$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

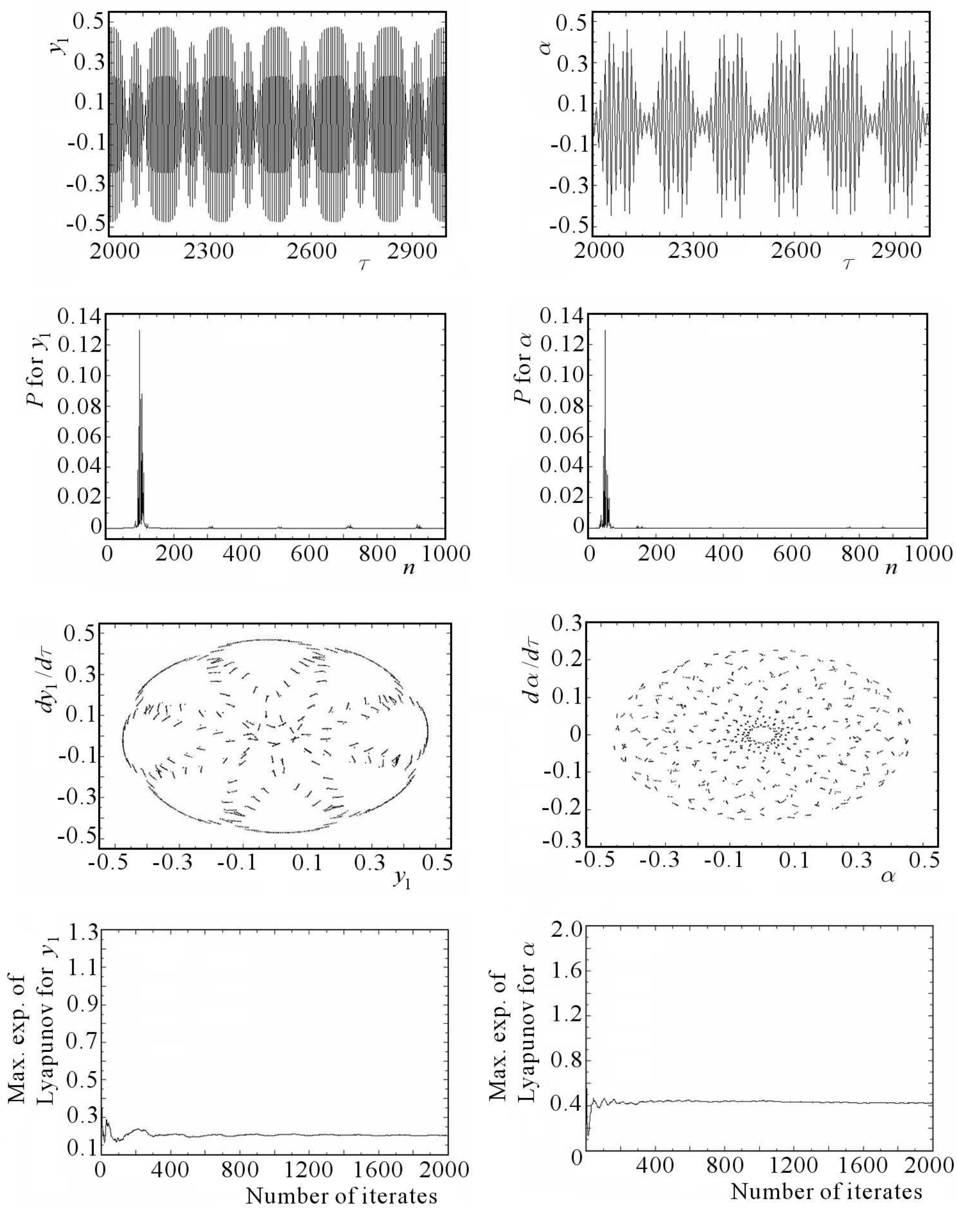


Fig. 6. Time histories, Power spectral densities, Poincaré's maps and maximum Lyapunov exponents corresponding to coordinates $y_1$ and $\alpha$ for the control parameter $u_1 = 1.504$ and for: $u_2 = 1.5$, $q = 0.2$, $\beta = 0.5$, $a_1 = 0.01$, $a_2 = 0.3$, $\gamma_1 = 0.001$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

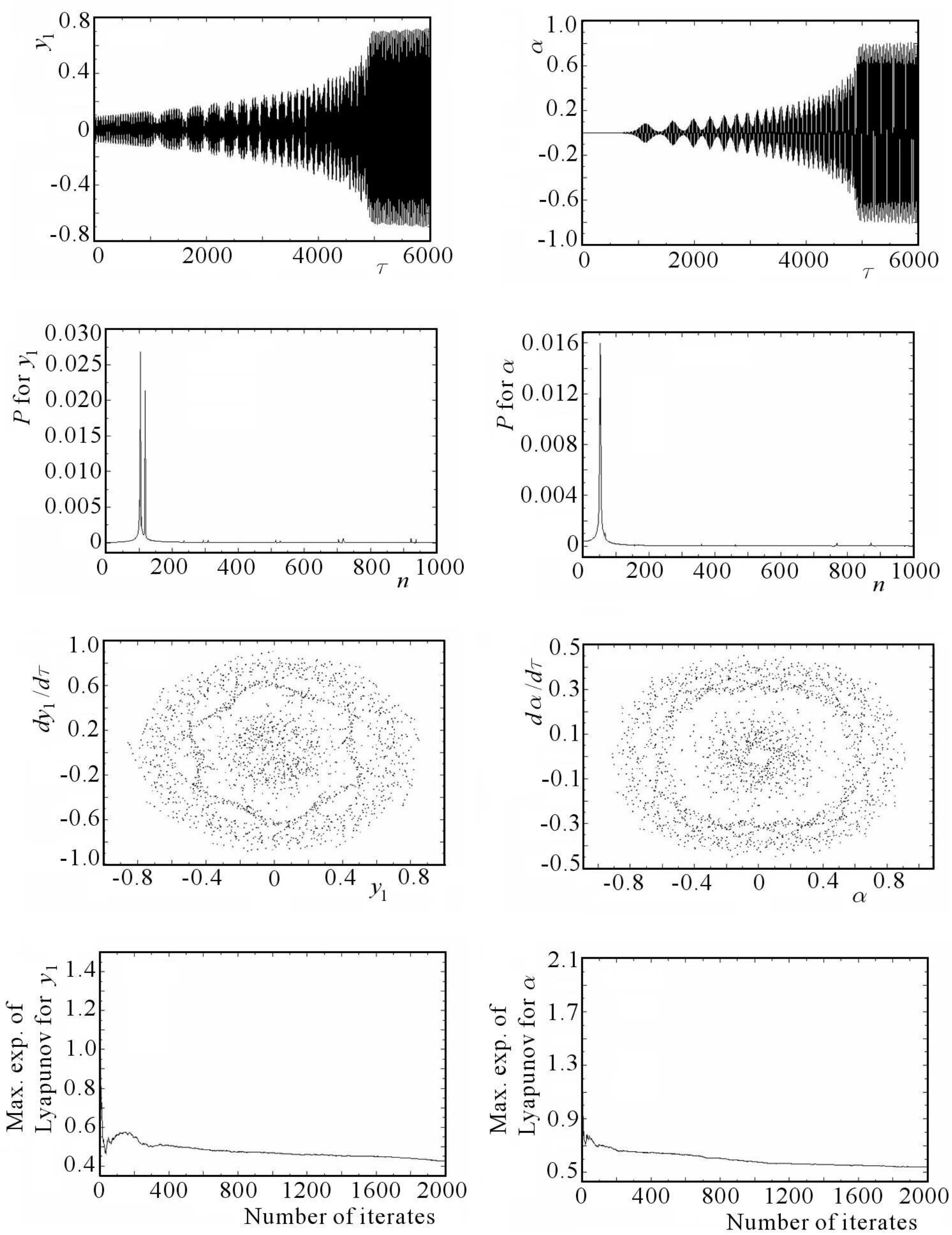


Fig. 7. Time histories, Power spectral densities, Poincaré's maps and maximum Lyapunov exponents corresponding to coordinate $\alpha$ for the control parameter $u_1 = 1.72$ and for: $u_2 = 1.5$, $q = 0.2$, $\beta = 0.5$, $a_1 = 0.01$, $a_2 = 0.3$, $\gamma_1 = 0.001$, $\gamma_2 = \gamma_3 = \gamma_4 = 0$

mics. Except for different kinds of periodic vibrations, also different kinds of irregular vibrations have been found. The bifurcation diagrams for the control parameter $u_1$, which is related to the voltage of the DC motor, showed for weaker damping many sudden qualitative changes, that is, many bifurcations in the chaotic atractors as well as in the periodic orbits. For every value of the control parameter $u_1$, these phenomena were verified in the phase space. The time histories, power spectral densities, Poincaré's maps and maximum Lyapunov exponents corresponding to coordinates of the system indicated a possibility of the onset of chaos. It has been shown that after a long time jumps in the amplitudes are possible. So this kind of vibrating systems should be investigated in an adequately long time to be sure that the results are correct.

In the future, we are going to continue the research using a non-linear characteristic of the source of power.

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## Drgania chaotyczne autoparametrycznego układu z nieidealnym źródłem energii

Streszczenie

W pracy uwzględniono wzajemne oddziaływania autoparametrycznego układu drgającego o dwóch stopniach swobody i układu wymuszającego, którym jest silnik elektryczny z niewyważoną masą o znanej charakterystyce. Układ podstawowy składa się z wahadła o długości $l$ i masie $m$ podwieszonego do ciała o masie $M$ zawieszonego na elemencie sprężystym. Uwzględniając nieidealne źródło energii dodaje się do badanego układu dodatkowy stopień swobody, bada się więc układ o trzech stopniach swobody, ale czas nie występuje w równaniach w postaci jawnej. Równania ruchu

rozwiązywano numerycznie i badano drgania w pobliżu rezonansu wewnętrznego i rezonansu zewnętrznego. W tym zakresie parametrów oprócz różnego rodzaju drgań regularnych mogą wystąpić również drgania chaotyczne. Charakter drgań nieregularnych weryfikowano analizując diagramy bifurkacyjne, przebiegi czasowe, transformaty Fouriera, mapy Poincaré oraz maksymalne wykładniki Lapunowa.

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