JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
46, 1, pp. 141-156, Warsaw 2008

ANALYSIS OF VIBRATION OF THREE-DEGREE-OF-FREEDOM DYNAMICAL SYSTEM WITH DOUBLE PENDULUM

Danuta Sado

Warsaw University of Technology, Poland
e-mail: dsado@poczta.onet.pl

Krzysztof Gajos

PEC Legionowo, Poland
e-mail: gajosy1@gazeta.pl

The nonlinear response of a three-degree-of-freedom vibratory system with a double pendulum in the neighborhood of internal and external resonances has been examined. Numerical and analytical methods have been applied for these investigations. Analytical solutions have been obtained by using the multiple scales method. This method is used to construct first-order non-linear ordinary differential equations governing the modulation of amplitudes and phases. Steady state solutions and their stability are computed for selected values of the system parameters.

Key words: nonlinear coupled oscillators, autoparametric vibrations, multiple scale method

1. Introduction

In complex three-degree-of-freedom vibrating systems with elements of pendulums suspended on a flexible element, the autoparametric excitation as a result of inertial coupling may occur (Sado, 1997). Dynamic systems of this kind with two degrees of freedom were widely discussed in the literature as autoparametric vibration eliminators (Bajaj and Johnson, 1990; Bajaj *et al.*, 1994; Banerjee *et al.*, 1996) or other structural components (Samaranayake and Bajaj, 1993; Sado, 2002; Shoeybi and Ghorashi, 2004). The effect of a parametric or autoparametric excitation on a three-mass system was studied by Tondl and Nabergoj (2004). Numerical simulations of a two mass system with three degrees of freedom with pendulums hanging down from a flexibly suspended body was investigated by Sado (2004) for an elastic pendulum and by Sado and Gajos (2003) for a double pendulum.

This paper describes the analytical solution of a three-degree-of-freedom system with a double pendulum. As it is a vibrating system with changing values of amplitudes and phases, in the analytical investigation the method of multiple scales was applied (Nayfeh and Mook, 1979). This method was used by several researchers (Ertas and Chew, 1990; Ji and Leung, 2003; Moon and Kang, 2003; Çevik and Pakdemirli, 2005; Rossikhin and Shitikova, 2006). Eliminating secular terms, we can observe conditions when the phenomenon of internal and external resonances is possible. Next, for the conditions of such resonances, steady-state solutions were investigated.

2. Equations of motion

The investigated system is shown in Fig. 1. The system consist of a double pendulum and a body of mass m_1 suspended on a flexible element of rigidity k, thus $S(y) = ky$. The pendulum of length l_1 and mass m_2 hangs down from

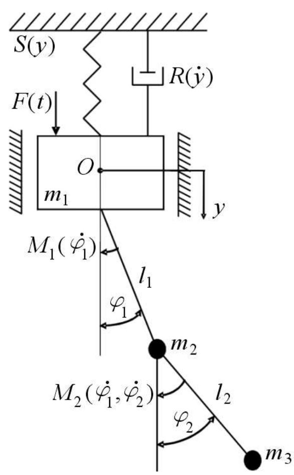

Fig. 1. Schematic diagram of the considered system

the body of mass m_1. The pendulum of length l_2 and mass m_3 is suspended on the body of mass m_2. It is assumed that a linear viscous damping force acts upon the body m_1 ($R(\dot{y}) = c_1\dot{y}$), and a linear damping momentum acts upon the pendulum of mass m_2 ($M_1(\dot{\varphi}_1) = c_2\dot{\varphi}_1$), and a linear damping momentum applied in the hinge opposes motion of the pendulum of mass m_3 $(M_2(\dot{\varphi}_1, \dot{\varphi}_2) = c_3(\dot{\varphi}_2 - \dot{\varphi}_1))$. The body of mass m_1 is subjected to a harmonic vertical excitation $F(t) = P_0 \cos \nu t$. This system has three degrees of freedom. As generalized coordinates, the vertical displacement y of the body of mass m_1 measured from the equilibrium position and the angles φ_1 and φ_2 of deflection of the pendulums measured from the vertical lines are assumed.

The equations of motion are derived as Lagrange's equations

$$
\begin{aligned}
&(m_1 + m_2 + m_3)\ddot{y} - l_1(m_2 + m_3)\ddot{\varphi}_1 \sin\varphi_1 - m_3 l_2 \ddot{\varphi}_2 \sin\varphi_2 + \\
&\qquad -(m_2 + m_3) l_1 \dot{\varphi}_1^2 \cos\varphi_1 - m_3 l_2 \dot{\varphi}_2^2 \cos\varphi_2 + ky + c_1\dot{y} = P_0 \cos\nu t \\
&-(m_2 + m_3)\ddot{y} \sin\varphi_1 + (m_2 + m_3) l_1 \ddot{\varphi}_1 + m_3 l_2 \ddot{\varphi}_2 \cos(\varphi_2 - \varphi_1) + \\
&\qquad -m_3 l_2 \dot{\varphi}_2^2 \sin(\varphi_2 - \varphi_1) + (m_2 + m_3) g \sin\varphi_1 + c_2\dot{\varphi}_1 - c_3(\dot{\varphi}_2 - \dot{\varphi}_1) = 0 \\
&-\ddot{y} \sin\varphi_2 + l_1 \ddot{\varphi}_1 \cos(\varphi_2 - \varphi_1) + l_2 \ddot{\varphi}_2 + l_1 \dot{\varphi}_1^2 \sin(\varphi_2 - \varphi_1) + \\
&\qquad +g \sin\varphi_2 + c_3(\dot{\varphi}_2 - \dot{\varphi}_1) = 0
\end{aligned}
\tag{2.1}
$$

Next, we introduce the dimensionless time $\tau = \omega_1 t$ and the following definitions

$$
\begin{array}{lll}
y_1 = \dfrac{y}{l_1} & y_{1st} = \dfrac{y_{st}}{l_1} & d_1 = \dfrac{m_2}{m_1} \\
d_2 = \dfrac{m_3}{m_1} & d_3 = \dfrac{d_1}{1 + d_1 + d_2} & d_4 = \dfrac{d_2}{1 + d_1 + d_2} \\
d_5 = d_3 + d_4 & d_6 = \dfrac{d_4}{d_3} & d_7 = 1 + d_6 \\
\omega_1^2 = \dfrac{k}{m_1 + m_2 + m_3} & \omega_2^2 = \dfrac{g}{l_1} & \omega_3^2 = \dfrac{g}{l_2} \\
c = \dfrac{l_2}{l_1} & \beta_1 = \dfrac{\omega_2}{\omega_1} & \gamma_1 = \dfrac{c_1}{m_2\omega_1} \\
\gamma_2 = \dfrac{c_2}{m_2 l_1^2 \omega_1} & \gamma_3 = \dfrac{c_3}{m_2 l_1^2 \omega_1} & \mu = \dfrac{\nu}{\omega_1} \\
p = \dfrac{P_0}{m_2 l_1 \omega_1^2} & &
\end{array}
\tag{2.2}
$$

3. The method of multiple scales

In order to find approximate solutions to equations of motion we use the method of multiple scales (Nayfeh and Mook, 1979). Partially, this problem for a system with a double pendulum was presented by Sado and Gajos (2005). For small oscillations, after transformations the equations of motion can be written down in the form

$$\begin{aligned}
&\ddot{y}_1 + y_1 - d_5\Big(\varphi_1 + \frac{\varphi_1^3}{6}\Big)\ddot{\varphi}_1 - d_4c\Big(\varphi_2 + \frac{\varphi_2^3}{6}\Big)\ddot{\varphi}_2 - \dot{\varphi}_1^2 d_5\Big(1 - \frac{\varphi_1^2}{4}\Big) + \\
&\qquad - d_4c\dot{\varphi}_2^2\Big(1 - \frac{\varphi_2^2}{4}\Big) = -d_3\gamma_1\dot{y}_1 + d_3p\cos(\mu\tau) \\
&d_5\ddot{\varphi}_1 - d_5\Big(\varphi_1 + \frac{\varphi_1^3}{6}\Big)\ddot{y}_1 + d_4c\Big(\varphi_1\varphi_2 + 1 - \frac{\varphi_2}{4} - \frac{\varphi_1}{4}\Big)\ddot{\varphi}_2 + \\
&\qquad + d_4c\dot{\varphi}_2^2\Big(\varphi_1 + \frac{\varphi_1^3}{6} - \varphi_1\frac{\varphi_2^2}{4} - \varphi_2 - \frac{\varphi_2^3}{6} - \varphi_2\frac{\varphi_1^2}{4}\Big) + d_5\beta_1^2\Big(\varphi_1 + \frac{\varphi_1^3}{6}\Big) + \\
&\qquad + d_3[\gamma_2\dot{\varphi}_1 - \gamma_3(\dot{\varphi}_2 - \dot{\varphi}_1)] = 0 \\
&c\ddot{\varphi}_2 - \Big(\varphi_2 + \frac{\varphi_2^3}{6}\Big)\ddot{y}_1 + \Big(\varphi_1\varphi_2 + 1 - \frac{\varphi_2^2}{4} - \frac{\varphi_1^2}{4}\Big)\ddot{\varphi}_1 - \dot{\varphi}_1^2\Big(\varphi_1 + \frac{\varphi_1^3}{6} - \varphi_1\frac{\varphi_2^2}{4} + \\
&\qquad - \varphi_2 - \frac{\varphi_2^3}{6} + \varphi_2\frac{\varphi_1^2}{4}\Big) + \beta_1^2\Big(\varphi_2 + \frac{\varphi_2^3}{6} - \frac{d_2c}{d_4}\gamma_2(\dot{\varphi}_2 - \dot{\varphi}_1) = 0
\end{aligned} \tag{3.1}$$

We introduce independent variables

$$\{T_0, T_1, T_2, \dots, T_n\} = \{\tau, \varepsilon\tau, \varepsilon^2\tau, \dots, \varepsilon^n\tau\} \tag{3.2}$$

and parameters

$$p_1 = \varepsilon^2\overline{p}_1 \qquad \gamma_1 = \varepsilon\overline{\gamma}_1 \qquad \gamma_2 = \varepsilon\overline{\gamma}_2 \qquad \gamma_3 = \varepsilon\overline{\gamma}_3 \tag{3.3}$$

Solutions to the dimensionless equations can be represented by

$$\begin{aligned}
y_1 &= \varepsilon y_{10} + \varepsilon^2 y_{11} + \dots \\
\varphi_1 &= \varepsilon\varphi_{10} + \varepsilon^2\varphi_{11} + \dots \\
\varphi_2 &= \varepsilon\varphi_{20} + \varepsilon^2\varphi_{21} + \dots
\end{aligned} \tag{3.4}$$

It follows that the derivatives with respect to τ become expansions in terms of partial derivatives with respect to T_n as

$$\begin{aligned}
\frac{d}{d\tau} &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \dots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots \\
\frac{d^2}{d\tau^2} &= \frac{\partial^2}{\partial T_0^2} + \varepsilon\frac{\partial^2}{\partial T_0\partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_0\partial T_2} + \varepsilon\frac{\partial^2}{\partial T_1\partial T_0} + \varepsilon^2\frac{\partial^2}{\partial T_1^2} + \varepsilon^2\frac{\partial^2}{\partial T_2\partial T_0} + \dots = \\
&= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1} + \varepsilon^2\Big(2\frac{\partial^2}{\partial T_0\partial T_2} + \frac{\partial^2}{\partial T_1^2}\Big) + \dots = \\
&= D_0^2 + 2\varepsilon D_0D_1 + \varepsilon^2(2D_0D_2 + D_1^2) + \dots
\end{aligned} \tag{3.5}$$

Substituting (3.3) and (3.4) into dimensionless equations (3.1) and equating the coefficients standing at ε^1 and ε^2 on both sides, we obtain:
— for ε^1

$$\begin{aligned}
&D_0^2 y_{10} + y_{10} = 0\\
&D_0^2\varphi_{10} - d_6\beta_1^2\varphi_{20} + d_7\beta_1^2\varphi_{10} = 0\\
&D_0^2\varphi_{20} - d_7\beta_2^2\varphi_{10} + d_7\beta_2^2\varphi_{20} = 0
\end{aligned} \tag{3.6}$$

— for ε^2

$$\begin{aligned}
&D_0^2 y_{11} + y_{11} = -2D_0D_1y_{10} + d_5(D_0\varphi_{10})^2 + d_4c(D_0\varphi_{20})^2 + d_3p\cos(\mu\tau) +\\
&\qquad -d_3\gamma_1D_0\varphi_{10} + 2d_6d_5\beta_1^2\varphi_{10}\varphi_{20} - d_6d_5\beta_1^2\varphi_{20}^2 - \frac{d_5^2}{d_3}\beta_1^2\varphi_{10}^2\\
&D_0^2\varphi_{11} - d_6\beta_1^2\varphi_{21} + d_7\beta_1^2\varphi_{11} = -2D_0D_1\varphi_{10} - d_7y_{10}\varphi_{10} + d_6y_{10}\varphi_{20} +\\
&\qquad -\Big(\frac{\gamma_3}{c} + \gamma_2 + \gamma_3\Big)D_0\varphi_{10} + \Big(\gamma_3 + \frac{\gamma_3}{c}\Big)D_0\\
&D_0^2\varphi_{21} - \frac{1}{c}(d_7\beta_1^2\varphi_{11} - d_7\beta_1^2\varphi_{21}) = -2D_0D_1\varphi_{20} + \frac{1}{c}(d_7y_{10}\varphi_{10} - d_7y_{10}\varphi_{20}) +\\
&\qquad -\frac{d_7\gamma_3}{c^2}(D_0\varphi_{20} - D_0\varphi_{10}) + \gamma_2D_0\varphi_{10} - \gamma_3(D_0\varphi_{20} - D_0\varphi_{10})
\end{aligned} \tag{3.7}$$

General solutions to equations (3.6) can be represented by

$$\begin{aligned}
&y_{10}(T_0,T_1,T_2) = A_1(T_1,T_2)\mathrm{e}^{\mathrm{i}\overline{\omega}_1T_0} + \overline{A}_1(T_1,T_2)\mathrm{e}^{-\mathrm{i}\overline{\omega}_1T_0}\\
&\varphi_{10}(T_0,T_1,T_2) = A_2(T_1,T_2)\mathrm{e}^{\mathrm{i}\overline{\omega}_2T_0} + \overline{A}_2(T_1,T_2)\mathrm{e}^{-\mathrm{i}\overline{\omega}_2T_0} +\\
&\qquad +A_3(T_1,T_2)\mathrm{e}^{\mathrm{i}\overline{\omega}_3T_0} + \overline{A}_3(T_1,T_2)\mathrm{e}^{-\mathrm{i}\overline{\omega}_3T_0}\\
&\varphi_{20}(T_0,T_1,T_2) = \varLambda_2A_2(T_1,T_2)\mathrm{e}^{\mathrm{i}\overline{\omega}_2T_0} + \overline{\varLambda}_2\overline{A}_2(T_1,T_2)\mathrm{e}^{-\mathrm{i}\overline{\omega}_2T_0} +\\
&\qquad +\varLambda_3A_3(T_1,T_2)\mathrm{e}^{\mathrm{i}\overline{\omega}_3T_0} + \overline{\varLambda}_3\overline{A}_3(T_1,T_2)\mathrm{e}^{-\mathrm{i}\overline{\omega}_3T_0}
\end{aligned} \tag{3.8}$$

We find natural frequencies of system (3.6) by substituting

$$y_1 = A_1\mathrm{e}^{\mathrm{i}\overline{\omega}T_0} + cc \qquad \varphi_1 = A_2\mathrm{e}^{\mathrm{i}\overline{\omega}T_0} + cc \qquad \varphi_2 = \varLambda A_2\mathrm{e}^{\mathrm{i}\overline{\omega}T_0} + cc \tag{3.9}$$

where cc represents the complex conjugate, and using the condition that the determinant of the matrix of coefficients is zero. In this case

$$\overline{\omega}_1 = 1$$

and

$$\overline{\omega}_{2,3}^2 = \frac{1}{2}\Big[-d_7\beta_1^2\Big(1 + \frac{1}{c}\Big) \pm \beta_1^2\sqrt{d_7^2\Big(1 + \frac{1}{c}\Big)^2 - \frac{4d_5d_6}{c}}\Big]$$

$$\Lambda_{2,3} = \frac{-\overline{\omega}_{2,3}^2 + d_7\beta_1^2}{d_6\beta_1^2}$$

Amplitudes and phases can be found by substituting (3.8) into (3.7). We obtain a system of equations

$$\begin{aligned}
D_0^2 y_{11} + y_{11} = {} & -2\mathrm{i}A_1'\mathrm{e}^{\mathrm{i}T_0} + d_5(-\overline{\omega}_2^2 A_2^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_2 T_0} - \overline{\omega}_3^2 A_3^2 e^{2\mathrm{i}\overline{\omega}_3 T_0} + \\
& +2\overline{\omega}_2\overline{\omega}_3\overline{A}_2 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_3-\overline{\omega}_2)T_0} - 2\overline{\omega}_2\overline{\omega}_3 A_2 A_3 \mathrm{e}^{\mathrm{i}(\omega_2+\omega_3)T_0} + 2\overline{\omega}_2^2 A_2\overline{A}_2 + 2\overline{\omega}_3^2 A_3\overline{A}_3) + \\
& +d_4 c(-\overline{\omega}_2^2 \Lambda_2^2 A_2^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_2 T_0} - \overline{\omega}_3^2 \Lambda_3^2 A_3^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_3 T_0} + 2\overline{\omega}_2\overline{\omega}_3 \overline{\Lambda}_2\overline{A}_2 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_3-\overline{\omega}_2)T_0} + \\
& -2\overline{\omega}_2\overline{\omega}_3 \Lambda_2 A_2 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_2+\overline{\omega}_3)T_0} + 2\overline{\omega}_2^2 \Lambda_2 A_2 \overline{\Lambda}_2\overline{A}_2 + 2\overline{\omega}_3^2 \Lambda_3 A_3 \overline{\Lambda}_3\overline{A}_3) + \\
& +\frac{1}{2} d_3 p \mathrm{e}^{\mathrm{i}\mu T_0} - d_3\gamma_1 \mathrm{i}\overline{\omega}_1 A_1 \mathrm{e}^{\mathrm{i}T_0} + 2d_5 d_6 \beta_1^2 [\Lambda_2 A_2^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_2 T_0} + \Lambda_3 A_3^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_3 T_0} + \\
& +(\Lambda_2 + \overline{\Lambda}_2) A_2\overline{A}_2 + (\Lambda_3 + \overline{\Lambda}_3) A_3\overline{A}_3 + (\Lambda_2 + \Lambda_3) A_2 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_2+\overline{\omega}_3)T_0} + \\
& +(\overline{\Lambda}_2 + \Lambda_3)\overline{A}_2 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_3-\overline{\omega}_2)T_0}] - d_5 d_6 \beta_1^2 (\Lambda_2^2 A_2^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_2 T_0} + \Lambda_3^2 A_3^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_3 T_0} + \\
& +2\overline{\Lambda}_2\overline{A}_2 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_3-\overline{\omega}_2)T_0} + 2\Lambda_2 A_2 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_2+\overline{\omega}_3)T_0} + 2\Lambda_2 A_2 \overline{\Lambda}_2\overline{A}_2 + \\
& +2\Lambda_3 A_3 \overline{\Lambda}_3\overline{A}_3) - \frac{d_5^2\beta_1^2}{d_3}(A_2^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_2 T_0} + A_3^2 \mathrm{e}^{2\mathrm{i}\overline{\omega}_3 T_0} + 2A_2\overline{A}_2 + \\
& +2A_3\overline{A}_3 + 2A_2 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_2+\overline{\omega}_3)T_0} + 2\overline{A}_2 A_3 \mathrm{e}^{\mathrm{i}(\overline{\omega}_3-\overline{\omega}_2)T_0})
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
D_0^2\varphi_{11} - d_6\beta_1^2\varphi_{21} + d_7\beta_1^2\varphi_{11} = {} & -2\mathrm{i}\overline{\omega}_2 A_2' \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} - 2\mathrm{i}\overline{\omega}_3 A_3' \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0} + \\
& -d_7(A_1 A_2 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_2)T_0} + A_1\overline{A}_2 \mathrm{e}^{\mathrm{i}(1-\overline{\omega}_2)T_0} + A_1 A_3 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_3)T_0} + \overline{A}_1 A_3 \mathrm{e}^{\mathrm{i}(-1+\overline{\omega}_3)T_0}) + \\
& +d_6(\Lambda_2 A_1 A_2 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_2)T_0} + \overline{\Lambda}_2 A_1\overline{A}_2 \mathrm{e}^{\mathrm{i}(1-\overline{\omega}_2)T_0} + \Lambda_3 A_1 A_3 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_3)T_0} + \\
& +\overline{\Lambda}_3\overline{A}_1 A_3 \mathrm{e}^{\mathrm{i}(-1+\overline{\omega}_3)T_0}) - \Big(\frac{\gamma_3}{c} + \gamma_2 + \gamma_3\Big)(\mathrm{i}\overline{\omega}_2 A_2 \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + \mathrm{i}\overline{\omega}_3 A_3 \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0}) + \\
& +\Big(\gamma_3 + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_2 \Lambda_2 A_2 \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + \mathrm{i}\overline{\omega}_3 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0})
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
D_0^2\varphi_{21} - \frac{d_7\beta_1^2}{c}\varphi_{11} + \frac{d_7\beta_1^2}{c}\varphi_{21} = {} & -2\mathrm{i}\overline{\omega}_2 \Lambda_2 A_2' \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} - 2\mathrm{i}\overline{\omega}_3 \Lambda_3 A_3' \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0} + \\
& -\frac{d_7}{c}(A_1 A_2 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_2)T_0} + A_1\overline{A}_2 \mathrm{e}^{\mathrm{i}(1-\overline{\omega}_2)T_0} + A_1 A_3 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_3)T_0} + \overline{A}_1 A_3 \mathrm{e}^{\mathrm{i}(-1+\overline{\omega}_3)T_0}) + \\
& -\frac{d_7}{c}(\Lambda_2 A_1 A_2 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_2)T_0} + \overline{\Lambda}_2 A_1\overline{A}_2 \mathrm{e}^{\mathrm{i}(1-\overline{\omega}_2)T_0} + \Lambda_3 A_1 A_3 \mathrm{e}^{\mathrm{i}(1+\overline{\omega}_3)T_0} + \\
& +\overline{\Lambda}_3\overline{A}_1 A_3 \mathrm{e}^{\mathrm{i}(-1+\overline{\omega}_3)T_0}) - \Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_2}{c} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_2 A_2 \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + \mathrm{i}\overline{\omega}_3 A_3 \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0}) + \\
& -\Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_2 \Lambda_2 A_2 \mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + \mathrm{i}\overline{\omega}_3 \Lambda_3 A_3 \mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0})
\end{aligned} \tag{3.12}$$

In this work, we analyze one combination of internal resonances and the external resonance

$$\mu = 1 \qquad 2\overline{\omega}_2 = 1 \qquad \overline{\omega}_3 = 3\overline{\omega}_2$$

We introduce detuning parameters σ_1, σ_2, σ_3 defined by

$$2\overline{\omega}_2 + \varepsilon\sigma_1 = 1 \qquad \overline{\omega}_3 = 3\overline{\omega}_2 + \varepsilon\sigma_2 \qquad \mu = 1 + \varepsilon\sigma_3 \tag{3.13}$$

Substituting (3.13) into equation (3.10) and eliminating terms that produce secular terms, we obtain

$$\begin{aligned}
&-2\mathrm{i}A_1' + 2\Big[d_5 + d_4 c\overline{\Lambda}_2\Lambda_3 + d_5 d_6\beta_1^2(\overline{\Lambda}_2 + \Lambda_3) - d_5 d_6\beta_1^2\overline{\Lambda}_2\Lambda_3 + \\
&-\frac{d_5^2}{d_3}\beta_1^2\Big]\overline{\omega}_2\overline{\omega}_3\overline{A}_2 A_3 \mathrm{e}^{\mathrm{i}T_1(-\sigma_1+\sigma_2)} - \Big(d_5 + d_4 c\Lambda_2^2 - 2d_5 d_6\beta_1^2\Lambda_2 + \\
&+d_5 d_6\beta_1^2\Lambda_2^2 + \frac{d_5^2}{d_3}\beta_1^2\Big)\overline{\omega}_2^2 A_2^2\mathrm{e}^{-\mathrm{i}T_1\sigma_1} + \frac{1}{2}d_3 p\mathrm{e}^{\mathrm{i}\sigma_3 T_1} - d_3\gamma_1\mathrm{i}\overline{\omega}_1 A_1 = 0
\end{aligned} \tag{3.14}$$

By introducing

$$A_1 = \frac{1}{2}a_1\mathrm{e}^{\mathrm{i}\alpha_1} \qquad A_2 = \frac{1}{2}a_2\mathrm{e}^{\mathrm{i}\alpha_2} \qquad A_3 = \frac{1}{2}a_3\mathrm{e}^{\mathrm{i}\alpha_3} \tag{3.15}$$

and

$$\begin{aligned}
&\theta_1 = 2\alpha_2 - \alpha_1 - T_1\sigma_1 \qquad \theta_2 = \alpha_3 - \alpha_2 - \alpha_1 - T_1\sigma_1 + T_1\sigma_2 \\
&\theta_3 = -\alpha_1 + T_1\sigma_3
\end{aligned} \tag{3.16}$$

into (3.14), we obtain the first modulation equation

$$-\mathrm{i}a_1' + a_1\alpha_1' + \frac{1}{4}f_1 a_2 a_3\mathrm{e}^{-\mathrm{i}\theta_2} + \frac{1}{4}f_2 a_2^2\mathrm{e}^{\mathrm{i}\theta_1} + \frac{1}{2}d_3 p\mathrm{e}^{\mathrm{i}\theta_3} - \frac{1}{2}\mathrm{i}d_3\gamma_1 a_1 = 0 \tag{3.17}$$

where

$$\begin{aligned}
f_1 &= 2d_5\overline{\omega}_2\overline{\omega}_3 + 2d_4 c\overline{\omega}_2\overline{\omega}_3\overline{\Lambda}_2\Lambda_3 + 2d_6 d_5\beta_1^2(\overline{\Lambda}_2 + \Lambda_3 - \overline{\Lambda}_2\Lambda_3) - \frac{2d_5^2\beta_1^2}{d_3} \\
f_2 &= -d_5\overline{\omega}_2^2 - d_4 c\overline{\omega}_2^2\Lambda_2^2 + d_5 d_6\beta_1^2(2\Lambda_2 - \Lambda_2^2) - \frac{d_5^2\beta_1^2}{d_3}
\end{aligned}$$

To determine the solvability conditions of (3.11) and (3.12), we seek for particular solutions in the form

$$\varphi_{11} = P_{11}\mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + P_{12}\mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0} \qquad \varphi_{21} = P_{21}\mathrm{e}^{\mathrm{i}\overline{\omega}_2 T_0} + P_{22}\mathrm{e}^{\mathrm{i}\overline{\omega}_3 T_0} \tag{3.18}$$

Substituting particular solutions (3.18) into equations (3.11), (3.12) and using resonant conditions (3.13) and equaling the coefficients of $\exp(\mathrm{i}\overline{\omega}_2 T_0)$ and $\exp(\mathrm{i}\overline{\omega}_3 T_0)$ on both sides, we obtain system of four equations

$$\begin{aligned} -\overline{\omega}_2^2 P_{11} - d_6\beta_1^2 P_{21} + d_7\beta_1^2 P_{11} &= R_{11} \\ -\overline{\omega}_2^2 P_{21} - \frac{d_7\beta_1^2}{c}(P_{11} + P_{21}) &= R_{21} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} -\overline{\omega}_3^2 P_{12} - d_6\beta_1^2 P_{22} + d_7\beta_1^2 P_{12} &= R_{12} \\ -\overline{\omega}_3^2 P_{22} - \frac{d_7\beta_1^2}{c}(P_{12} + P_{22}) &= R_{22} \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} R_{11} = & -2\mathrm{i}\overline{\omega}_2 A_2' - d_7(A_1\overline{A}_2\mathrm{e}^{\mathrm{i}T_1\sigma_1} + \overline{A}_1 A_3\mathrm{e}^{\mathrm{i}T_1(\sigma_2-\sigma_1)}) + \\ & + d_6(A_1\overline{A}_2\overline{\varLambda}_2\mathrm{e}^{\mathrm{i}T_1\sigma_1} + \overline{A}_1 A_3\varLambda_3\mathrm{e}^{\mathrm{i}T_1(\sigma_2-\sigma_1)}) + \\ & - \Big(\frac{\gamma_3}{c} + \gamma_2 + \gamma_3\Big)(\mathrm{i}\overline{\omega}_2 A_2 + \Big(\frac{\gamma_3}{c} + \gamma_3\Big)(\mathrm{i}\overline{\omega}_2 A_2\varLambda_2 \\ R_{21} = & -2\mathrm{i}\overline{\omega}_2 A_2'\delta\varLambda_2 - d_7(A_1\overline{A}_2\mathrm{e}^{\mathrm{i}T_1\sigma_1} + \overline{A}_1 A_3\mathrm{e}^{\mathrm{i}T_1(\sigma_2-\sigma_1)}) + \\ & + \frac{d_7}{c}(A_1\overline{A}_2\overline{\varLambda}_2\mathrm{e}^{\mathrm{i}T_1\sigma_1} + \overline{A}_1 A_3\varLambda_3\mathrm{e}^{\mathrm{i}T_1(\sigma_2-\sigma_1)}) + \\ & + \Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_2}{c} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_2 A_2 - \Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_2 A_2\varLambda_2 \\ R_{12} = & -2\mathrm{i}\overline{\omega}_3 A_3' - d_7 A_1 A_2\mathrm{e}^{\mathrm{i}T_1(-\sigma_2+\sigma_1)} + d_6 A_1 A_2\varLambda_2\mathrm{e}^{\mathrm{i}T_1(-\sigma_2+\sigma_1)}) + \\ & + \Big(\frac{\gamma_3}{c} + \gamma_2 + \gamma_3\Big)(\mathrm{i}\overline{\omega}_3 A_3 + \Big(\frac{\gamma_3}{c} + \gamma_3\Big)(\mathrm{i}\overline{\omega}_3 A_3\varLambda_3 \\ R_{22} = & -2\mathrm{i}\overline{\omega}_3 A_3'\varLambda_3 + d_7 A_1 A_2\mathrm{e}^{\mathrm{i}T_1(-\sigma_2+\sigma_1)} - \frac{d_7}{c}A_1 A_2\varLambda_2\mathrm{e}^{\mathrm{i}T_1(-\sigma_2+\sigma_1)}) + \\ & + \Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_2}{c} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_3 A_3 - \Big(\frac{d_7\gamma_3}{c^2} + \frac{\gamma_3}{c}\Big)(\mathrm{i}\overline{\omega}_3 A_3\varLambda_3 \end{aligned}$$

We reduce the problem of determination of the solvability conditions of equations (3.11), (3.12) to finding solvability conditions of equations (3.19) and (3.20).The determinants of the coefficient matrices of equations (3.19) and (3.20) are the same and equal 0 according to conditions on the natural frequencies of system (3.6).

Then the solvability conditions are

$$\begin{vmatrix} R_{11} & -d_6\beta_1^2 \\ R_{21} & -\overline{\omega}_2^2 + \dfrac{d_7\beta_1^2}{c} \end{vmatrix} = 0 \tag{3.21}$$

for equations (3.19) and

$$\begin{vmatrix} R_{12} & -d_6\beta_1^2 \\ R_{22} & -\overline{\omega}_3^2 + \dfrac{d_7\beta_1^2}{c} \end{vmatrix} = 0 \tag{3.22}$$

for equations (3.20).

Substituting (3.15) and (3.16) and after some transformations, we obtain two modulation equations

$$\begin{aligned} &-\mathrm{i}a_2' + a_2\alpha_2' + \frac{f_4}{4f_3\overline{\omega}_2}a_1a_2\mathrm{e}^{-\mathrm{i}\theta_1} + \frac{f_5}{4f_3\overline{\omega}_2}a_1a_3\mathrm{e}^{\mathrm{i}\theta_2} + \frac{f_6}{2f_3}\mathrm{i}a_2 = 0 \\ &-\mathrm{i}a_3' + a_3\alpha_3' + \frac{f_8}{4f_7\overline{\omega}_3}a_1a_2\mathrm{e}^{-\mathrm{i}\theta_2} + \frac{f_9}{2f_7}\mathrm{i}a_3 = 0 \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} f_3 &= -\overline{\omega}_2^2 + \frac{d_7\beta_1^2}{c} + d_6\Lambda_2\beta_1^2 \\ f_4 &= \Big(-\overline{\omega}_2^2 + \frac{d_7\beta_1^2}{c}\Big)(-d_7 + d_6\overline{\Lambda}_2) + \frac{d_6d_7\beta_1^2}{c}(1 - \overline{\Lambda}_2) \\ f_5 &= \Big(-\overline{\omega}_2^2 + \frac{d_7\beta_1^2}{c}\Big)(-d_7 + d_6\Lambda_3) + \frac{d_6d_7\beta_1^2}{c}(1 - \Lambda_3) \\ f_6 &= \Big(-\overline{\omega}_2^2 + \frac{d_7\beta_1^2}{c}\Big)\Big[-\frac{\gamma_3}{c} - \gamma_3 - \gamma_2 + \Lambda_2\Big(\frac{\gamma_3}{c} + \gamma_3\Big)\Big] + \\ &\quad + \frac{d_6\beta_1^2}{c}\Big[\frac{d_7\gamma_3}{c} + \gamma_3 + \gamma_2 - \Lambda_2\Big(\frac{d_7\gamma_3}{c} + \gamma_3\Big)\Big] \\ f_7 &= -\overline{\omega}_3^2 + \frac{d_7\beta_1^2}{c} + d_6\Lambda_3\beta_1^2 \\ f_8 &= \Big(-\overline{\omega}_3^2 + \frac{d_7\beta_1^2}{c}\Big)(-d_7 + d_6\Lambda_2) + \frac{d_6d_7\beta_1^2}{c}(1 - \Lambda_2) \\ f_9 &= \Big(-\overline{\omega}_3^2 + \frac{d_7\beta_1^2}{c}\Big)\Big[-\frac{\gamma_3}{c} - \gamma_3 - \gamma_2 + \Lambda_3\Big(\frac{\gamma_3}{c} + \gamma_3\Big)\Big] + \\ &\quad + \frac{d_6\beta_1^2}{c}\Big[\frac{d_7\gamma_3}{c} + \gamma_3 + \gamma_2 - \Lambda_3\Big(\frac{d_7\gamma_3}{c} + \gamma_3\Big)\Big] \end{aligned} \tag{3.24}$$

To separate the real and imaginary parts of modulation equations (3.17), (3.23) and (3.24), we have to transform $\exp(\mathrm{i}\theta)$ into a complex form $\exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$. We obtain six modulation equations

$$\begin{aligned}
a_1' &= a_2a_3f_1\sin\theta_2 + a_2^2f_2\sin\theta_1 + \frac{1}{2}d_3p\sin\theta_3 - \frac{1}{2}d_3\gamma_1a_1\\
a_1\alpha_1' &= -a_2a_3f_1\cos\theta_2 - a_2^2f_2\cos\theta_1 + \frac{1}{2}d_3p\cos\theta_3\\
a_2' &= -\frac{f_4}{4f_3\overline{\omega}_2}a_1a_2\sin\theta_1 + \frac{f_5}{4f_3\overline{\omega}_2}a_1a_3\sin\theta_2 + \frac{f_6}{2f_3}a_2\\
a_2\alpha_2' &= -\frac{f_4}{4f_3\overline{\omega}_2}a_1a_2\cos\theta_1 - \frac{f_5}{4f_3\overline{\omega}_2}a_1a_3\cos\theta_2\\
a_3' &= -\frac{f_8}{4f_7\overline{\omega}_3}a_1a_2\sin\theta_2 + \frac{f_9}{2f_7}a_3\\
a_3\alpha_3' &= -\frac{f_8}{4f_7\overline{\omega}_3}a_1a_2\cos\theta_2
\end{aligned} \tag{3.25}$$

From these equations, we look for steady-state motion. In this case, we have

$$\begin{aligned}
a_1' &= 0 \qquad & a_2' &= 0 \qquad & a_3' &= 0\\
\theta_1' &= 0 \qquad & \theta_2' &= 0 \qquad & \theta_3' &= 0
\end{aligned} \tag{3.26}$$

We obtain a system of equations

$$\begin{aligned}
&a_2a_3f_1\sin\theta_2 + a_2^2f_2\sin\theta_1 + \frac{1}{2}d_3p\sin\theta_3 - \frac{1}{2}d_3\gamma_1a_1 = 0\\
&-a_2a_3f_1\cos\theta_2 - a_2^2f_2\cos\theta_1 + \frac{1}{2}d_3p\cos\theta_3 - a_1\sigma_3 = 0\\
&-\frac{f_4}{4f_3\overline{\omega}_2}a_1a_2\sin\theta_1 + \frac{f_5}{4f_3\overline{\omega}_2}a_1a_3\sin\theta_2 + \frac{f_6}{2f_3}a_2 = 0\\
&-\frac{f_4}{4f_3\overline{\omega}_2}a_1a_2\cos\theta_1 - \frac{f_5}{4f_3\overline{\omega}_2}a_1a_3\cos\theta_2 - a_2\frac{\sigma_3+\sigma_1}{2} = 0\\
&-\frac{f_8}{4f_7\overline{\omega}_3}a_1a_2\sin\theta_2 + \frac{f_9}{2f_7}a_3 = 0\\
&-\frac{f_8}{4f_7\overline{\omega}_3}a_1a_2\cos\theta_2 - a_3\frac{3\sigma_3-2\sigma_2+3\sigma_1}{2} = 0
\end{aligned} \tag{3.27}$$

After transformations, we get amplitude equations

$$
\begin{aligned}
&f_8^2a_1^2a_2^2 - 4[f_9^2 + f_7^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)]\overline{\omega}_3^2a_3^2 = 0\\
&f_4^2f_8^2a_1^2a_2^4 - (2f_5f_9\overline{\omega}_3a_3^2 + 2f_6f_8\overline{\omega}_2a_2^2)^2 +\\
&\quad -[2f_5f_7\overline{\omega}_3(3\sigma_3 - 2\sigma_2 + 3\sigma_1)a_3^2 - 2f_3f_8\overline{\omega}_2(\sigma_3 + \sigma_1)a_2^2]^2 = 0\\
&f_4^2f_8^2d_3^2p^2a_1^2 - [4f_9\overline{\omega}_3(f_4f_1 + f_5f_2)a_3^2 + 4f_6f_8f_9\overline{\omega}_2a_2^2 - f_4f_8d_3\gamma_1a_1^2]^2 +\\
&\quad +[4f_7\overline{\omega}_3(f_4f_1 - f_5f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)a_3^2 + 4f_3f_8f_2\overline{\omega}_2(\sigma_3 + \sigma_1)a_2^2 +\\
&\quad -f_4f_8\sigma_3a_1^2]^2 = 0
\end{aligned}
\tag{3.28}
$$

From equations (3.28), we obtain

$$
a_2^4\Big[-\frac{h_1^2h_4}{h_3^2}a_1^4 + \Big(h_3 - \frac{h_6h_1}{h_3}\Big)a_1^2 - h_5\Big] = 0 \tag{3.29}
$$

We have two types of solutions, and these possibilities are examined in turn:
— case I – one-frequency solution

$$
a_2 = 0 \qquad \text{then} \qquad a_3 = 0 \quad \text{and} \quad a_1^2(h_{10}a_1^2 - h_7) = 0
$$

so

$$
a_1 = 0 \qquad \text{or} \qquad a_1 = \sqrt{\frac{h_7}{h_{10}}} \tag{3.30}
$$

— case II – multi-frequency solution

$$
\frac{h_1^2h_4}{h_3^2}a_1^4 - \Big(h_3 - \frac{h_6h_1}{h_3}\Big)a_1^2 + h_5 = 0 \tag{3.31}
$$

so

$$
a_1 = \sqrt{\frac{h_3 - \frac{h_6h_1}{h_3} \pm \sqrt{\varDelta_1}}{\frac{2h_1^2h_4}{h_3^2}}} \tag{3.32}
$$

where

$$
\varDelta_1 = \Big(h_3 - \frac{h_6h_1}{h_3}\Big)^2 - 4\frac{h_4h_5h_1^2}{h_3^2}
$$

and from (3.28)

$$
a_2 = \sqrt{\frac{-\Big(\frac{h_{13}h_1}{h_3}a_1^4 + h_{11}a_1^2\Big) \pm \sqrt{\varDelta_2}}{2\Big(\frac{h_1^2h_8}{h_3^2}a_1^4 + \frac{h_{12}h_1}{h_3}a_1^2 + h_9\Big)}} \qquad a_3 = \sqrt{\frac{h_1}{h_3}}\,a_1a_2 \tag{3.33}
$$

where

$$\varDelta_2 = \Big(\frac{h_{13}h_1}{h_3}a_1^4 + h_{11}a_1^2\Big)^2 - 4\Big(\frac{h_1^2h_8}{h_3^2}a_1^4 + \frac{h_{12}h_1}{h_3}a_1^2 + h_9\Big)(h_{10}a_1^4 - h_7a_1^2)$$

and

$$\begin{aligned}
&h_1 = f_8^2 \qquad\qquad h_2 = 4\overline{\omega}_3^2(f_9^2 + f_7^2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2\\
&h_3 = f_4^2f_8^2 \qquad\qquad h_4 = 4f_5^2\overline{\omega}_3^2[f_9^2 + f_7^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2]\\
&h_5 = 4f_8^2\overline{\omega}_2^2[f_6^2 + f_3^2(\sigma_3 + \sigma_1)^2]\\
&h_6 = 8f_5f_8\overline{\omega}_2\overline{\omega}_3[f_6f_9 - f_3f_7(3\sigma_3 - 2\sigma_2 + 3\sigma_1)(\sigma_3 + \sigma_1)\\
&h_7 = f_4^2f_8^2d_3^2p^2 \qquad\qquad h_{10} = f_4^2f_8^2(d_3^2\gamma_1^2 + \sigma_3^2)\\
&h_8 = 16f_9^2\overline{\omega}_3^2(f_4f_1 + f_5f_2)^2 + 16f_7^2\overline{\omega}_3^2(f_4f_1 - f_5f_2)^2(3\sigma_3 - 2\sigma_2 + 3\sigma_1)^2\\
&h_9 = 16f_6^2f_8^2f_9^2\overline{\omega}_2^2 + 16f_2^2f_3^2f_8^2\overline{\omega}_2^2(\sigma_3 + \sigma_1)^2\\
&h_{11} = -8f_4f_8^2\overline{\omega}_2[f_6f_9d_3\gamma_1 + f_2f_3\sigma_3(\sigma_3 + \sigma_1)]\\
&h_{12} = 32f_8\overline{\omega}_2\overline{\omega}_3[f_6f_9^2(f_4f_1 + f_5f_2) +\\
&\qquad + f_1f_3f_7(f_4f_1 - f_5f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)(\sigma_3 + \sigma_1)]\\
&h_{13} = -8f_4f_8\overline{\omega}_3[f_9d_3\gamma_1(f_4f_1 + f_5f_2) + f_7\sigma_3(f_4f_1 - f_5f_2)(3\sigma_3 - 2\sigma_2 + 3\sigma_1)]
\end{aligned}$$

Both cases of solutions (one-frequency and multi-frequency) are presented in Figs. 2-5. In Fig. 2 and Fig. 3 amplitudes a_1, a_2, a_3 are plotted as functions of the amplitude of excitation p. We can see the jump phenomenon associated with the varying amplitude p. We have regions where two of the three solutions are stable. The initial conditions determine which of these solutions gives the response. We can clearly see the saturation phenomenon, when the amplitude a_1 assumes its maximum value for stable solutions.

In Fig. 4 and Fig. 5, these amplitudes are presented versus the detuning parameter σ_1. We can see the jump phenomenon associated with the varying frequency ω_1 according with the amplitude a_1.

4. Conclusions

The multiple scales method can be used to find an approximate solution for a system with three degrees of freedom with variable amplitudes and phases. We can find resonance conditions (sometimes the resonance area is very narrow

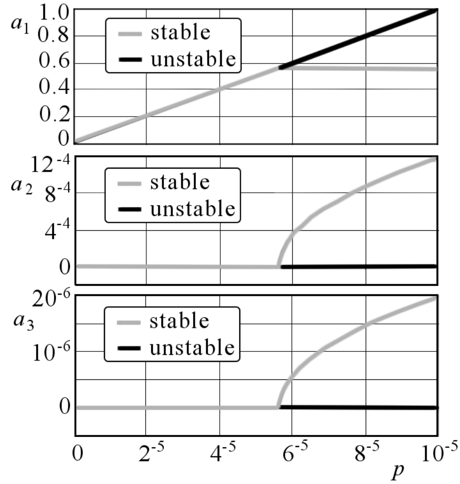

Fig. 2. Amplitudes of the response as functions of the amplitude of the excitation; $d_1 = 0.9$, $d_2 = 1.6$, $c = 1$, $\beta_1 = 0.67082$, $\mu = 1$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.00001$, $\gamma_3 = 0.00001$, $\sigma_1 = \sigma_2 = \sigma_3 = 0$

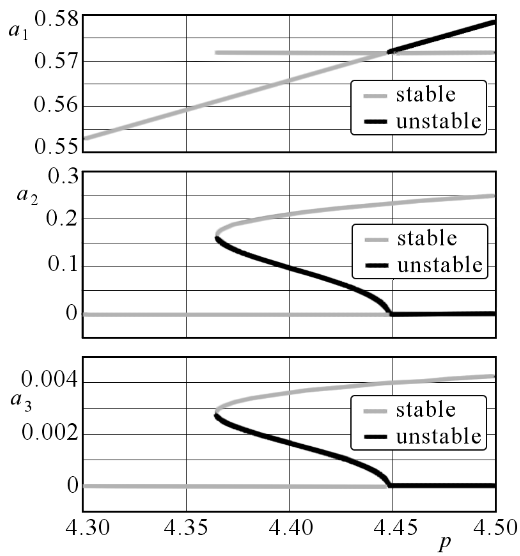

Fig. 3. Amplitudes of the response as functions of the amplitude of the excitation; $d_1 = 0.9$, $d_2 = 1.6$, $c = 1$, $\beta_1 = 0.67082$, $\mu = 1$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.00001$, $\gamma_3 = 0.00001$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$

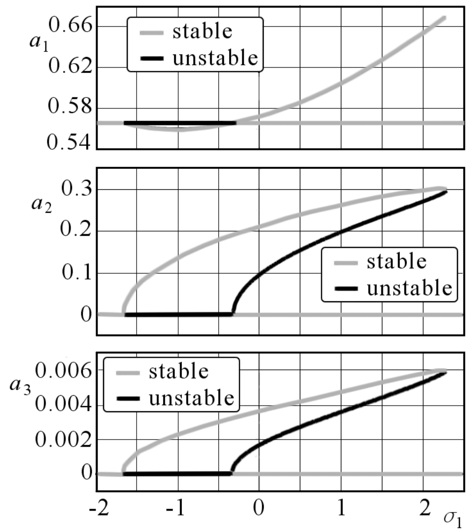

Fig. 4. Frequency-response curves; $d_1 = 0.9$, $d_2 = 1.6$, $c = 1$, $\beta_1 = 0.67082$, $\mu = 1$, $p = 4.4$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.00001$, $\gamma_3 = 0.00001$, $\sigma_2 = 0$, $\sigma_3 = 1$

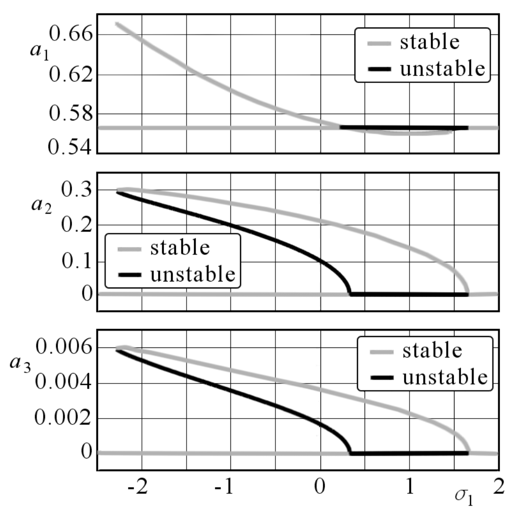

Fig. 5. Frequency-response curves; $d_1 = 0.9$, $d_2 = 1.6$, $c = 1$, $\beta_1 = 0.67082$, $\mu = 1$, $p = 4.4$, $\gamma_1 = 0.0001$, $\gamma_2 = 0.00001$, $\gamma_3 = 0.00001$, $\sigma_2 = 0$, $\sigma_3 = -1$

and difficult to find numerically). It is possible to investigate steady state solutions for different combinations of external and internal resonances. We can observe regions where the solutions are stable or unstable, and can clearly see the saturation phenomenon.

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Analiza drgań dynamicznego układu z podwójnym wahadłem o trzech stopniach swobody

Streszczenie

W pracy przebadano drgania nieliniowego układu o trzech stopniach swobody z podwójnym wahadłem w otoczeniu rezonansów wewnętrznych i zewnętrznych. Badania przeprowadzono analitycznie i numerycznie. Rozwiązanie analityczne uzyskano przy użyciu metody wielu skali czasowych. Metoda posłużyła do zbudowania nieliniowych równań różniczkowych pierwszego rzędu opisujących modulację amplitud i faz. Rozwiązanie ustalone i jego stabilność zostały przedstawione dla wybranych wartości parametrów układu.

Manuscript received February 21, 2007; accepted for print April 4, 2007