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# ON THE ANALYTICAL SOLUTION FOR THE SLEWING FLEXIBLE BEAM-LIKE SYSTEM: ANALYSIS OF THE LINEAR PART OF THE PERTURBED PROBLEM

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The dynamical system investigated in this work is a nonlinear flexible beam-like structure in slewing motion. Non-dimensional and perturbed governing equations of motion are presented. The analytical solution for the linear part of these perturbed equations for ideal and for non-ideal cases are obtained. This solution is necessary for the investigation of the complete weak nonlinear problem where all nonlinearities are small perturbations around a linear known solution. This investigation shall help the analyst in the modelling of dynamical systems with structure-actuator interactions.

*Key words:* non-ideal system, flexible structure, perturbed equations, analytical solution

## 1. Introduction

The study of dynamic behaviour (and control) of slewing flexible structures has in view the improvement of lightweight and faster structures. These investigations are complex and present continuing interest from researchers and scientists.

The applications of the theory of slewing flexible structures may be divided basically into two groups: robotics and aerospace structure applications. Typically, the modal analysis approach is the most popular model approach to aerospace slewing structures, and the finite element approach is the one most frequently used to investigate robotic manipulators. Low inherent damping, small natural frequencies, and extreme light weights are some common characteristics of these systems, and which make them vulnerable to any external/internal disturbances (such as angular maneuvers, impacts, etc). Robot arms with such characteristics are easy to carry out, need smaller actuators and can reach objectives in a greater workspace since they are thinner and longer than the rigid ones usually used for the same task. DC motors are popular actuators for lightweight manipulators not only because they can generate a wide range of torque and angular velocity, but also because they are quiet, clean and efficient. Nowadays, in the competitive world, the search for such kind of mechanical systems is an increasing preoccupation.

Many of the published papers in this area are concerned with the dynamics and/or control of flexible beam-like structures. Little effort has been focused on the actuator-structure interaction. This interaction affects the dynamics of the whole slewing system (actuator included).

The goal of this work is to deal with this interaction. In systems such as lightweight robotic manipulators, solar panels and antennas in satellites, helicopter blades and so on, the mutual interaction between the angular displacement of the slewing axis and the flexible structure deflection can be very important in high angular speed maneuvers (Balthazar *et al.*, 1999, 2001-2004).

Dynamical systems, in which this interaction occurs, are called non-ideal systems (Kononenko, 1969). In the ideal system, there is no mutual interaction and only the actuator dynamics excites the structure dynamics.

Governing equations of motion for the ideal and non-ideal damped beam-like slewing flexible structure connected to a DC motor are presented with discussions on nonlinear effects in Fenili (2004), Fenili *et al.* (2004). These equations and boundary conditions are nondimensional and scaled quantities also known as perturbed equations of motion because of the small parameter which multiplies nonlinear and damping terms (Fenili, 2000; Fenili and Balthazar, 2005a,b). Some experimental results and discussions were related in Fenili *et al.* (2001).

Next, some papers are mentioned from the current literature concerning the subject treated here, in order to understand better the position of this problem in the main current literature.

## 2. A brief literature review

Slewing flexible structures were first considered, in literature, by Book *et al.* (1975). In that paper, the authors applied a modal truncated model on a system composed by two flexible beams with two joints, and included discussion on the controller design with a torque source. Considerable efforts have been performed since then by other authors in this direction. Some of those authors are mentioned next (without undeserving many others not mentioned) in this paper.

Cannon and Schmidt (1984) developed and demonstrated stable and precise position control of one end of a very flexible beam by using direct measurements from the free end position as a basis for torquing at the other end. The authors demonstrated that a satisfactory feedback tip-control response can be achieved with a good dynamic model of a flexible arm. Bayo (1987) analyzed a structural finite element algorithm on the linear Bernoulli-Euler beam in order to calculate the end torque necessary to produce desired motion at the free tip of a flexible link. The computed torque, smaller than the one required for the rigid link of some weight, provided desired tip motion without overshoot. The authors used this approach in both open loop and feedback controls. Juang *et al.* (1986) studied a slewing flexible structure experimentally; Juang and Horta (1987) developed a hardware set-up to study slewing control for slewing flexible structures and using linear optimal control to design active control (implemented in an analog computer); Juang *et al.* (1989) discussed several important issues related to slewing experiments with flexible structures including nonlinearities and calibration of actuators and sensors. In Hamilton *et al.* (1991) a simple experimental set-up was made up by a pair of single-axis flexible beams attached to a DC servo motor in order to illustrate collocated and noncollated control for this kind of structures. An optical encoder and strain gauges provided hub and beam position information, respectively. Their results on non-collocated zero-placement control illustrate that the feedback from the beam and motor hub provides the necessary information for vibration suppression in slewing maneuvers. Yang *et al.*) (1994) realized numerical simulations concerning slewing control tasks for a planar articulated double-beam structure. Garcia and Inman (1990) investigated the modeling of a single link flexible slewing beam torque driven by a DC motor at the slewing axis. Effects of modal participation factors in the slewing equations of motion were discussed. These factors were indicative of the degree into which actuators and flexible structures interacted with one another dynamically. Boundary conditions of the slewing beam were determined by the actuator that drove the entire system. Garcia and Inman (1991) considered the slewing control of an active flexible structure by examining the governing equations of motion of an integrated actuator-structure system composed by a thin aluminum beam torque driven by an armature controlled electric motor and actuated by a piecewise distributed piezoceramic actuator. Their approach offered an advantage of reducing the peak voltage demands on the motor. In addition, that active structure approach substantially reduced the maximum tip deflection of the beam. Sah *et al.* (1993), using a finite element algorithm, considered the closed-loop performance and the dynamic interaction between a DC motor and a slewing beam. The authors concluded that systems with an appropriate amount of actuator-beam interaction tend to be more easily controlled and require a modest amount of actuator efforts, but systems with little actuator-beam interaction are especially prone to beam vibrations and require feedback of beam dynamics for good closed-loop performance. It was also shown that systems with excessive interaction require stabilization efforts in order to obtain good transient performance and tend to consume increased levels of actuation energy. In Kwak *et al.* (1994), a slewing flexible beam equipped with piezoelectric sensors and actuators was modeled via extended Hamilton's principle considering nonlinearities coming from rigid body rotational motion, whereas the elastic vibration was assumed to be small compared to rigid body rotation. The authors used a decentralized control technique in which the control of the whole system was divided into the slewing control and the vibration suppression control. The sliding model control was proposed as the slewing control, and the modal space positive feedback plus disturbance-counteracting control was developed for suppression of vibrations. These techniques were verified by experiments. The experimental results showed that the decentralized control performs satisfactorily, but high frequency vibrations remained uncontrolled. In this case, there will be a need for more actuators, sensors and more powerful piezoelectric actuators.

Finally, Fenili (2000) has started the study of nonlinear dynamic behaviour of flexible structures in planar slewing motions for linear and nonlinear curvature assumptions. A schematic of the system investigated there (and here) is depicted in Fig. 1. In this paper, the authors present a search for an approximate analytical solution to this problem. A short version of this subject was discussed in Fenili and Balthazar (2005a). An experimental apparatus was constructed in order to verify the theoretical developments and is shown in Fig. 2 (Fenili *et al.*, 2001).

This paper is organized as follows. In Section 3 the governing equations of motion are presented. In Section 4 the analytical solution of the linear non-

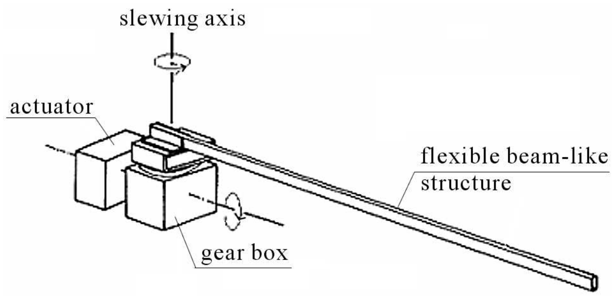


Fig. 1. Schematic of a slewing beam-like flexible structure (Fenili, 2000)

Fig. 2. Experimental set-up of the slewing flexible beam-like structure (Fenili, 2000; Fenili *et al.*, 2001)

ideal problem (one mode expansion) is developed. In Section 5 some concluding remarks are presented. In Section 6 some acknowledgements are made. Finally, main bibliographic references used in this work and two appendixes in order to detail and/or give values for the expressions, equations and coefficients presented along the paper are listed.

## 3. Governing equations of motion

The governing equations of motion for the system depicted in Fig. 1 are derived from the extended Hamilton principle (Fenili, 2000; Fenili *et al.*, 2004a). This mathematical model is based on a nonlinear curvature assumption for the beam. The nonlinear equations obtained are put on a dimensionless form and scaled in order to become perturbed equations of the kind (Nayfeh and Mook, 1973)

$$\text{linear terms } + \epsilon^2(\text{nonlinear terms + damping}) = 0 \tag{3.1}$$

The small parameter, $\epsilon$, in (3.1) is given by

$$\epsilon = \frac{r^2}{L^2} \tag{3.2}$$

where $r$ is the radius of gyration of the cross-section area of the beam and $L$ is the length of the beam. The perturbed equations are then discretized through the assumed method of modes. On this assumption, the deflection variable, $v(x,t)$, is written as

$$v(x,t) = \sum_{i=1}^{n} \phi_i(x) q_i(t) \tag{3.3}$$

where $n$ is the number of modes, $\phi_i(x)$ is any admissible function which represents the assumed solutions in space and $q_i(t)$ are unknown solutions in time. Considering the longitudinal deflection of the beam, $u(x,t)$, as being of the order $O(\epsilon^2)$, assuming $\theta$ as the slewing angle and $i_a$ as the armature current in the DC motor, the perturbed discretized governing equations of motion for the non-ideal system, after some manipulations in order to eliminate u(x,t) from the set of governing equations (Fenili, 2000), are given by

$$\begin{aligned}
&\dot{i}_a + c_1 i_a + c_2\dot{\theta} = c_1 U \\
&\ddot{\theta} + c_3\dot{\theta} - c_4 i_a - c_5\phi_1''(0)q_1 = 0 \\
&\ddot{q}_1 + w_1^2 q_1 + \alpha_1\ddot{\theta} + \epsilon^2[\mu\dot{q}_1 + \beta_{11}\dot{\theta}^2 q_1 - \wp_{111}\dot{\theta}q_1\dot{q}_1 + \lambda_{111}\ddot{\theta}q_1^2 + \\
&\quad +\mathit{\Lambda}_{1111}q_1\dot{q}_1^2 + \mathit{\Lambda}_{1111}q_1^2\ddot{q}_1 + \mathit{\Gamma}_{1111}q_1^3] = 0
\end{aligned} \tag{3.4}$$

The boundary conditions are given by

$$\begin{aligned}
\phi(0,t) &= 0 \qquad & \phi'(0,t) &= 0 \\
\phi''(1,t) &= 0 \qquad & \phi'''(1,t) &= 0
\end{aligned}$$

The coefficients of Eqs. (3.4) are presented in Appendix A. For the linear ideal problem, one must consider $c_5 = 0$ in Eq. $(3.4)_2$.

## 4. Analytical solution to the linear problem (one mode expansion)

For the linear system ($\epsilon = 0$ in Eq. $(3.4)_3$) associated with the primary resonance of the first flexural mode, one needs to make $U = 0$ in Eq. $(3.4)_1$.

To run numerical simulations, the initial condition is given in the variable $q_1$ (associated to the time behaviour of the beam deflection), as can be seen in Figures 3 to 5.

Considering one mode expansion $(n = 1)$, the set of equations to be analytically solved is given by

$$
\begin{aligned}
&\dot{i}_a + c_1 i_a + c_2 \dot{\theta} = 0 \\
&\ddot{\theta} + c_3 \dot{\theta} - c_4 i_a - c_5 \phi_1''(0) q_1 = 0 \\
&\ddot{q}_1 + w_1^2 q_1 + \alpha_1 \ddot{\theta} = 0
\end{aligned}
\tag{4.1}
$$

The mode shapes considered, are given by

$$
\phi_i(x) = \cosh(a_i x) - \cos(a_i x) - \alpha_i[\sinh(a_i x) - \sin(a_i x)]
$$

where

$$
\alpha_i = \frac{\cosh(a_i L) + \cos(a_i L)}{\sinh(a_i L) + \sin(a_i L)}
$$

and $a_i$ is associated with the eigenvalues of the clamped-free undamped and not excited linear (Bernoulli-Euler) beam.

### 4.1. Analytical solution for $q_1(t)$

Dividing Eq. $(4.1)_3$ by $-\alpha_1$, adding to Eq. $(4.1)_2$ and solving for $i_a$, gives

$$
i_a = -\Big(\frac{1}{\alpha_1 c_4}\Big)\ddot{q}_1 - \Big(\frac{w_1^2}{\alpha_1 c_4} + \frac{c_5 \phi_i''(0)}{c_4}\Big) q_1 + \Big(\frac{c_3}{c_4}\Big)\dot{\theta}
\tag{4.2}
$$

Differentiating (4.2) with respect to time, yields

$$
\dot{i}_a = -\Big(\frac{1}{\alpha_1 c_4}\Big)\dddot{q}_1 - \Big(\frac{w_1^2}{\alpha_1 c_4} + \frac{c_5 \phi_i''(0)}{c_4}\Big) \dot{q}_1 + \Big(\frac{c_3}{c_4}\Big)\ddot{\theta}
\tag{4.3}
$$

Substituting Eqs. (4.2) and (4.3) into $(4.1)_1$, results

$$
\begin{aligned}
&-\Big(\frac{1}{\alpha_1 c_4}\Big)\dddot{q}_1 - \Big(\frac{c_1}{\alpha_1 c_4}\Big)\ddot{q}_1 - \Big(\frac{w_1^2}{\alpha_1 c_4} + \frac{c_5 \phi_i''(0)}{c_4}\Big)\dot{q}_1 - \Big(\frac{c_1 w_1^2}{\alpha_1 c_4} + \frac{c_1 c_5 \phi_i''(0)}{c_4}\Big) q_1 + \\
&+\Big(\frac{c_3}{c_4}\Big)\ddot{\theta} + \Big(\frac{c_1 c_3}{c_4} + c_2\Big)\dot{\theta} = 0
\end{aligned}
\tag{4.4}
$$

Solving Eq. $(4.1)_3$ for $\ddot{\theta}$, gives

$$
\ddot{\theta} = -\Big(\frac{1}{\alpha_1}\Big)\ddot{q}_1 - \Big(\frac{w_1^2}{\alpha_1}\Big) q_1
\tag{4.5}
$$

Integrating Eq. (4.5) from $0$ to $t$, are obtains:

$$\dot{\theta} = -\Big(\frac{1}{\alpha_1}\Big)\dot{q}_1 - \Big(\frac{w_1^2}{\alpha_1}\Big)\int\limits_0^t q_1\ dt \tag{4.6}$$

Substituting Eqs. $(4.1)_{1,2}$ into (4.4), differentiating the resulting equation with respect to time and multiplying by $-1$, results

$$A\ddot{\ddot{q}}_1 + B_1 + C\ddot{q}_1 + D\dot{q}_1 + Eq_1 = 0 \tag{4.7}$$

where:

$$A = \frac{1}{\alpha_1 c_4} \qquad B = \frac{c_1 + c_3}{\alpha_1 c_4}$$

$$C = \frac{w_1^2 + c_1 c_3}{\alpha_1 c_4} + \frac{c_5\phi_1''(0)}{c_4} + \frac{c_2}{\alpha_1} \qquad D = \frac{(c_1 + c_3)w_1^2}{\alpha_1 c_4} + \frac{c_1 c_5 \phi_1''(0)}{c_4}$$

$$E = \frac{c_1 c_3 w_1^2}{\alpha_1 c_4} + \frac{c_2 w_1^2}{\alpha_1}$$

Equation (4.7) is a fourth order ordinary differential equation with constant coefficients in the variable $q_1$. The characteristic equation associated to (4.7) is given by

$$Ar^4 + Br^3 + Cr^2 + Dr + E = 0 \tag{4.8}$$

If Eq. (4.8) has only real and not equal roots, a solution to (4.7) is of the form

$$q_1 = C_1 \mathrm{e}^{r_{1a}t} + C_2 \mathrm{e}^{r_{2a}t} + C_3 \mathrm{e}^{r_{3a}t} + C_4 \mathrm{e}^{r_{4a}t} \tag{4.9}$$

If Eq. (4.8) has only complex roots, the solution to (4.7) is of the form

$$q_1 = C_1 \mathrm{e}^{r_{1a}t}\cos(r_{1b}t) + C_2 \mathrm{e}^{r_{2a}t}\sin(r_{2b}t) + C_3 \mathrm{e}^{r_{3a}t}\cos(r_{3b}t) + C_4 \mathrm{e}^{r_{4a}t}\sin(r_{4b}t) \tag{4.10}$$

If Eq. (4.8) has two complex and two real roots, the solution to (4.7) is of the form

$$q_1 = C_1 \mathrm{e}^{r_{1a}t}\cos(r_{1b}t) + C_2 \mathrm{e}^{r_{2a}t}\sin(r_{2b}t) + C_3 \mathrm{e}^{r_{3a}t} + C_4 \mathrm{e}^{r_{4a}t} \tag{4.11}$$

where $r_{ia}$ and rib are associated which each of the possible complex roots of (4.8).

Let the initial time be represented by $t_r$. Equations (4.1) can be rewritten in $t_r$, resulting

$$\begin{aligned} \dot{i}_{ar} &= -c_1 i_{ar} - c_2\dot{\theta}_r \\ \ddot{\theta}_r &= -c_3\dot{\theta}_r + c_4 i_{ar} + c_5\phi_i''(0)q_{1r} \\ \ddot{q}_{1r} &= -w_1^2 q_{1r} - \alpha_1\ddot{\theta}_r \end{aligned} \tag{4.12}$$

Differentiating $(4.12)_{2,3}$ with respect to time, one finds $\dddot{\theta}_r$ and $\dddot{q}_{1r}$.

Using Eqs. (4.9), (4.10) or (4.11) and their time derivatives (in $t = t_r$), a set of algebraic equations can be formulated. From these equations, one can find the constants $C_i$. With the values in Appendix B, one can easily find $A$, $B$, $C$, $D$ and $E$, and calculate roots of the characteristic equations. The solution given by Eq. (4.11), for each case, is given by

$$\begin{aligned}
q_{1_{case1}} &= 1.1958 \cdot 10^4 \mathrm{e}^{-24.7510t} \cos(962.4100t) + 345.9672 \mathrm{e}^{-24.7510t} \cdot \\
&\quad \cdot \sin(962.4100t) + 60.3107 \mathrm{e}^{-613.4746t} + 2.0655 \mathrm{e}^{-6.4628 \cdot 10^{-5} t} \\
q_{1_{case2}} &= -11.4037 \mathrm{e}^{-47.1299t} \cos(78.5155t) + 4.9359 \mathrm{e}^{-47.1299t} \sin(78.5155t) - \\
&\quad -0.0433 \mathrm{e}^{-592.6730t} + 8.4131 \cdot 10^4 \mathrm{e}^{-0.0113t} \qquad (4.13) \\
q_{1_{case3}} &= 7.8895 \cdot 10^5 \mathrm{e}^{-0.5874t} \cos(0.8169t) - 6.3616 \cdot 10^5 \mathrm{e}^{-0.5874t} \sin(0.8169t) + \\
&\quad +15.2815 \mathrm{e}^{-591.4816t} - 0.1039 \cdot 10^5 \mathrm{e}^{-95.4528t}
\end{aligned}$$

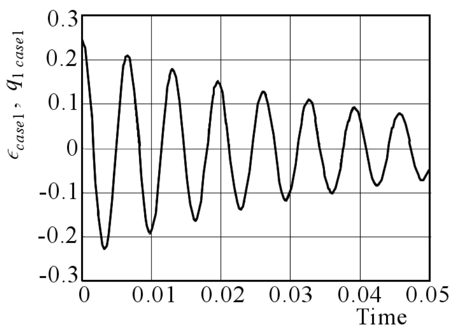


Fig. 3. Non-dimensional transverse deflection of the versus time – case 1

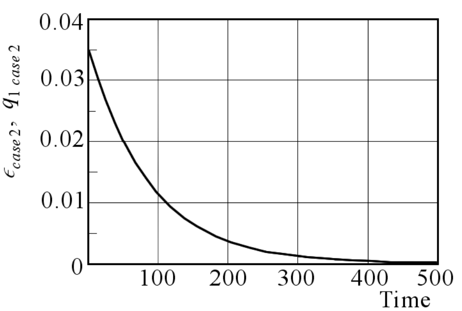


Fig. 4. Non-dimensional transverse deflection of the versus time – case 2

Figures 3 to 5 illustrate solutions (4.13). These solutions are stable and damped.

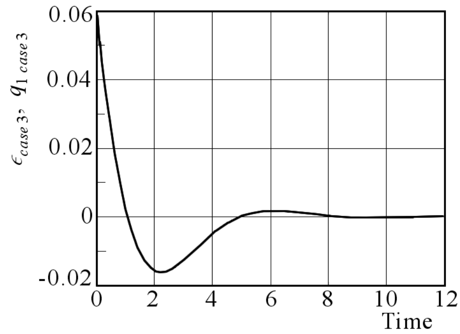


Fig. 5. Non-dimensional transverse deflection of the versus time – case 3

## 5. Analytical solution for $\theta(t)$

Integrating Eq. (4.6) in time and using solution (4.11), are obtains

$$\begin{aligned}\theta = E_1 \mathrm{e}^{r_{1a}t}\cos(r_{1b}t) + E_2\mathrm{e}^{r_{1a}t}\sin(r_{1b}t) + E_3\mathrm{e}^{r_{2a}t}\cos(r_{2b}t) + \\ +E_4\mathrm{e}^{r_{2a}t}\sin(r_{2b}t) + E_5\mathrm{e}^{r_{3a}t} + E_6\mathrm{e}^{r_{4a}t} + E_7 t + E_8\end{aligned} \tag{5.1}$$

where

$$E_0 = \frac{C_1 w_1^2(r_{1a}^2 - r_{1b}^2)}{\alpha_1(r_{1a}^2 + r_{1b}^2)^2} \qquad E_1 = -\frac{C_1 + E_0\alpha_1}{\alpha_1}$$

$$E_2 = -\frac{2C_1 w_1^2 r_{1a} r_{1b}}{\alpha_1(r_{1a}^2 + r_{1b}^2)^2} \qquad E_3 = -\frac{2C_2 w_1^2 r_{2a} r_{2b}}{\alpha_1(r_{2a}^2 + r_{2b}^2)^2}$$

$$E_4 = -\frac{C_2((r_{2a}^2 + r_{2b}^2)^2 + w_1^2(r_{2a}^2 - r_{2b}^2))}{\alpha_1(r_{2a}^2 + r_{2b}^2)^2}$$

$$E_5 = -\frac{C_3(r_{3a}^2 + w_1^2)}{\alpha_1 r_{3a}^2} \qquad E_6 = -\frac{C_4(r_{4a}^2 + w_1^2)}{\alpha_1 r_{4a}^2}$$

$$E_7 = \frac{w_1^2}{\alpha_1}\Big(\frac{C_1 r_{1a}}{r_{1a}^2 + r_{1b}^2} + \frac{C_3}{r_{3a}} + \frac{C_4}{r_{4a}} - \frac{C_2 r_{2b}}{r_{2a}^2 + r_{2b}^2}\Big)$$

$$E_8 = \frac{w_1^2}{\alpha_1}\Big(\frac{C_3}{r_{3a}^2} + \frac{C_4}{r_{4a}^2} + \frac{E_3\alpha_1}{w_1^2}\Big) + E_0$$

Figures 6 to 8 show solution (4.15).

For case 1, the angular displacement is damped. For cases 2 and 3, the oscillation of the beam is damped but the angular displacement increases in time.

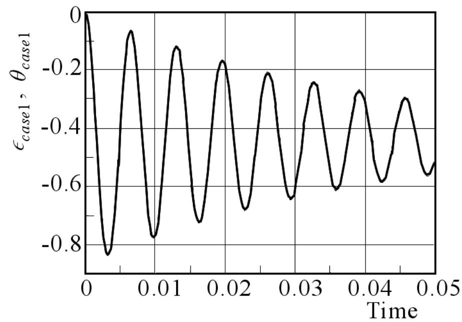

Fig. 6. Angular displacement (nondimensional) – case 1

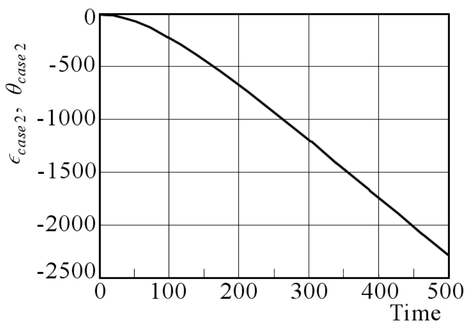

Fig. 7. Angular displacement (nondimensional) – case 2

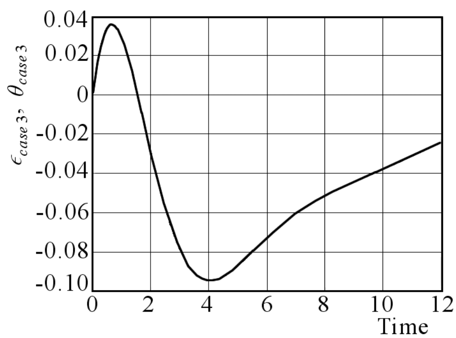

Fig. 8. Angular displacement (nondimensional) – case 3

### 5.1. Analytical solution for $i_a(t)$

Substituting $q_1$ (Eq. (4.11)), its second time derivative and $\dot{\theta}$ (using Eq. (4.2)), gives

$$
\begin{aligned}
i_a = {} & (H_1 \cos(r_{1b}t) + H_2 \sin(r_{1b}t))\mathrm{e}^{r_{1a}t} + [H_3 \sin(r_{2b}t) + H_4 \cos(r_{2b})]\mathrm{e}^{r_{2a}t} + \\
& + H_5 \mathrm{e}^{r_{3a}t} + H_6 \mathrm{e}^{r_{4a}t} + H_7 + H_8
\end{aligned}
\tag{5.2}
$$

where

$$H_1 = -\frac{C_1(r_{1a}^2 - r_{1b}^2 + w_1^2)}{\alpha_1 c_4} - \frac{C_1 c_{50} + c_3 A(E_1 r_{1a} + E_2 r_{1b})}{c_4}$$

$$c_{50} = c_5 \phi_i''(0) \qquad H_2 = \frac{2C_1 r_{1a} r_{1b}}{\alpha_1 c_4} - H_{2b}$$

$$H_{2b} = \frac{c_3 A(E_1 r_{1b} - E_2 r_{1a})}{c_4}$$

$$H_3 = -\frac{C_2(r_{2a}^2 - r_{2b}^2 + w_1^2)}{\alpha_1 c_4} - \frac{C_2 c_{50} - c_3 B(E_3 r_{2b} - E_4 r_{2a})}{c_4}$$

$$H_4 = -\frac{2C_2 r_{2a} r_{2b}}{\alpha_1 c_4} + \frac{c_3 B(E_3 r_{2a} + E_4 r_{2b})}{c_4}$$

$$H_5 = H_{5b} - \frac{C_3 c_{50} + c_3 E_5 r_{3a}}{c_4}$$

$$H_{5b} = -\frac{C_3(r_{3a}^2 + w_1^2)}{\alpha_1 c_4}$$

$$H_6 = -\frac{C_4(r_{4a}^2 + w_1^2)}{\alpha_1 c_4} - \frac{C_4 c_{50} + c_3 E_6 r_{4a}}{c_4} \qquad H_7 = \frac{c_3 E_7}{c_4}$$

$$H_8 = -H_1 - H_4 - H_5 - H_6 - H_7$$

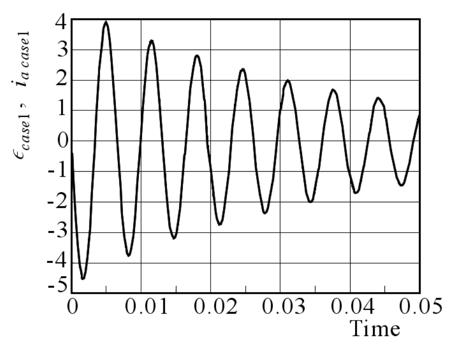


Fig. 9. Armature current (nondimensional) – case 1

The constant $H_8$ is introduced in order to guarantee that $i_a(0) = 0$. Figures 9 to 11 illustrate the solution given by Eq. (4.16) to each of the studied cases.

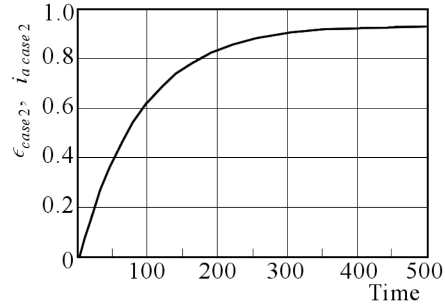


Fig. 10. Armature current (nondimensional) – case 2

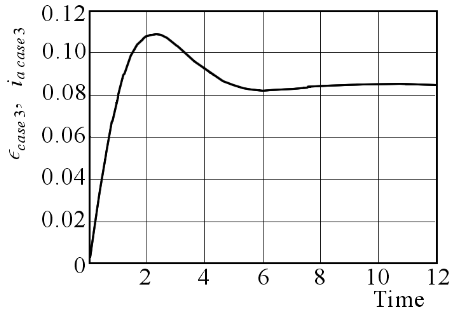


Fig. 11. Armature current (nondimensional) – case 3

## 6. Conclusions

The analytical solution to the set of differential equations associated with the linear part of the perturbed governing equations of motion for the general problem under investigation (nonlinear slewing flexible beam-like structures) is presented here. This solution, considered of the order $\epsilon^0$, is very important to start the search for a perturbed solution to the weak nonlinear problem presented in Eqs. (3.4). The analytical solutions obtained here for the actuator and for the beam (and plotted in Fig. 3 to 11) are stable.

The next step in this investigation consists in applying a perturbation technique to solve the perturbed problem using the solution presented here (a remarkable task!). Using the multiple scale method, for example, the nonlinear perturbed equations of motion of the order $\epsilon^2$ to be solved now are given by

$$\begin{aligned}
&\frac{\partial i_{a0}}{\partial T_0} + c_1 i_{a0} + c_2 \frac{\partial \theta_0}{\partial T_0} + \epsilon^2 \Big[\frac{\partial i_{a1}}{\partial T_0} + \frac{\partial i_{a0}}{\partial T_1} + c_1 i_{a1} + c_2 \frac{\partial \theta_1}{\partial T_0} + c_2 \frac{\partial \theta_0}{\partial T_1}\Big] = \\
&= \epsilon^2 \Big[\frac{uc_1}{2i}(\mathrm{e}^{\mathrm{i}\Omega t} - \mathrm{e}^{-\mathrm{i}\Omega t})\Big]
\end{aligned}$$

$$\frac{\partial^2\theta_0}{\partial T_0^2} + c_3\frac{\partial\theta_0}{\partial T_0} - c_4 i_{a0} - c_5\phi_1''(0)q_{10} +$$
$$+\epsilon^2\Big[\frac{\partial^2\theta_1}{\partial T_0^2} + c_3\frac{\partial\theta_1}{\partial T_0} - c_4 i_{a1} - c_5\phi_1''(0)q_{11} + c_3\frac{\partial\theta_0}{\partial T_1} + 2\frac{\partial\theta_0}{\partial T_0\partial T_1}\Big] = 0$$

$$\frac{\partial^2 q_{10}}{\partial T_0^2} + w_1^2 q_{10} + \alpha_1\frac{\partial^2\theta_0}{\partial T_0^2} + \epsilon^2\Big[\frac{\partial^2 q_{11}}{\partial T_0^2} + w_1^2 q_{11} + \alpha_1\frac{\partial^2\theta_1}{\partial T_0^2} + \beta_{11}q_{10}\Big(\frac{\partial\theta_0}{\partial T_0}\Big)^2 +$$
$$-\wp_{111}q_{10}\frac{\partial\theta_0}{\partial T_0}\frac{\partial q_{10}}{\partial T_0} + \lambda_{111}q_{10}^2\frac{\partial^2\theta_0}{\partial T_0^2} + \Lambda_{1111}q_{10}\Big(\frac{\partial q_{10}}{\partial T_0}\Big)^2 +$$
$$+\Lambda_{1111}q_{10}^2\frac{\partial^2 q_{10}}{\partial T_0^2} + \Gamma_{1111}q_{10}^3 + 2\alpha_1\frac{\partial\theta_0}{\partial T_0\partial T_1} + 2\frac{\partial q_{10}}{\partial T_0\partial T_1}\Big] = 0$$

## A. The coefficients of equations (3.4)

$$c_1 = \frac{R_a T}{L_a} \qquad c_2 = N_g \qquad c_3 = \frac{c_v N_g^2 T}{I_t}$$

$$c_4 = \frac{N_g K_t K_b T^2}{L_m I_t} \qquad c_5 = \frac{EIT^2}{LI_t} \qquad I_t = I_{shaft} + N_g^2 I_{motor}$$

$$R_{ij} = \int\limits_0^x \phi_i'(\xi)\phi_j'(\xi)\;d\xi = R_{ji} \qquad V_i = -\int\limits_x^1 \phi_i(\xi)\;d\xi$$

$$\Lambda_{ijk\ell} = \int\limits_0^1 (S_{jk}\phi_i''\phi_\ell + R_{jk}\phi_i'\phi_\ell)\;dx \qquad \alpha_\ell = \int\limits_0^1 x\phi_\ell\;dx$$

$$S_{ij} = -\int\limits_x^1 \Big[\int\limits_0^\eta \phi_i'(\xi)\phi_j'(\xi)\;d\xi\Big]\;d\eta$$

$$W_{ij} = -\int\limits_x^1 \phi_i'(\xi)\phi_j(\xi)\;d\xi$$

$$\wp_{ij\ell} = 2\int\limits_0^1 (R_{ij}\phi_\ell - \phi_i''V_j\phi_\ell - \phi_i'\phi_j\phi_\ell)\;dx$$

$$\beta_{i\ell} = \Big[\int\limits_0^1 \Big(x\phi_i'\phi_\ell + \frac{1}{2}(x^2-1)\phi_i''\phi_\ell\Big)\;dx\Big] - 1$$

$$\lambda_{ij\ell} = \int_0^1 \Big(-\frac{1}{2}R_{ij}\phi_\ell + \phi_i''V_j\phi_\ell + \phi_i'\phi_j\phi_\ell\Big)\,dx$$

$$\Gamma_{ijk\ell} = \int_0^1 \Big[\frac{3}{1.8780^4}\phi_i'\phi_j''\phi_k'''\phi_\ell + \frac{3}{2(1.8780^4)}\phi_i''\phi_j''\phi''\phi_\ell +$$
$$+w_j^2(\phi_i'\phi_j\phi_k'\phi_\ell + W_{ij}\phi_k''\phi_\ell)\Big]\,dx$$

where: $R_a$ represents the armature resistance, $T$ is the period of the first natural frequency of the beam, $L_a$ – armature inductance, $c_v$ – motor internal damping, $I_{shaft}$ – inertia of the connecting motor-beam shaft, $I_{motor}$ – inertia of the motor, $K_t$ – torque constant, $K_b$ – back e.m.f. constant, $E$ – Young's modulus, $I$ – inertia of the beam cross section around the neutral axis, $L$ – beam length, $\phi_\ell$ – each of the flexural vibration mode and $w_\ell$ – corresponding model frequency.

## B. Beam properties (cases 1 to 3) and motor parameters

$T_1 = 1$ $\qquad w_1 = 1$ $\qquad I_{eixo} = 0.0000369\,\mathrm{kg\,m^2}$

$\alpha_1 = 0.570157$ $\qquad \phi_i''(0) = 7.05377$

### B.1. Beam

**Case 1:** aluminum, $L = 0.20\,\mathrm{m}$, $E = 0.70 \cdot 10^{11}\,\mathrm{N/m^2}$, height $= 0.02544\,\mathrm{m}$, height $= 0.02544\,\mathrm{m}$, $A = 0.00008039\,\mathrm{m^2}$, $I = 6.6896 \cdot 10^{-11}\,\mathrm{m^4}$, $\rho = 2700\,\mathrm{kg/m^3}$, $\epsilon = 0.000020804$

**Case 2:** aluminum, $L = 1.40\,\mathrm{m}$, $E = 0.70 \cdot 10^{11}\,\mathrm{N/m^2}$, height $= 0.02544\,\mathrm{m}$, basis $= 0.00316\,\mathrm{m}$, $A = 0.00008039\,\mathrm{m^2}$, $I = 6.6896 \cdot 10^{-11}\,\mathrm{m^4}$, $\rho = 2700\,\mathrm{kg/m^3}$, $\epsilon = 0.00000042456$

**Case 3:** steel, $L = 0.8720\,\mathrm{m}$, $E = 2.10 \cdot 10^{11}\,\mathrm{N/m^2}$, height $= 0.01587\,\mathrm{m}$, basis $= 0.00082\,\mathrm{m}$, $A = 0.00001301\,\mathrm{m^2}$, $I = 7.2918 \cdot 10^{-13}\,\mathrm{m^4}$, $\rho = 7800\,\mathrm{kg/m^3}$, $\epsilon = 0.000000073710$

Obs.: "height" and "basis" refer to the beam cross section (Fenili,2000).

**B.2. DC motor parameters**

$C_m = 0.0046290\,\mathrm{Nms/rad}$, $K_t = 0.0528140\,\mathrm{Nm/A}$, $R_a = 1.9149520\,\Omega$, $K_b = 0.0528140\,\mathrm{Vs/rad}$, $L_m = 0.0031000\,\mathrm{H}$, $I_{motor} = 0.0000654\,\mathrm{kg\,m^2}$.

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**O analitycznym rozwiązaniu dla układu podatnej belki obrotowej: analiza części liniowej zagadnienia perturbowanego**

Streszczenie

Układem dynamicznym badanym w pracy jest podzespół zawierający podatną belkę poruszającą się ruchem obrotowym w płaszczyźnie poziomej. Zaprezentowano bezwymiarowe i perturbowane równania ruchu. Rozwiązania analityczne części zlinearyzowanej tych równań uzyskano dla przypadku idealnego i nieidealnego. Taka postać rozwiązania jest niezbędna do analizy zupełnego i słabo nieliniowego problemu, w którym nieliniowości stanowią niewielkie perturbacje wokół rozwiązania liniowego. Prezentowane wyniki badań mogą okazać się pomocą dla analityków zajmujących się modelowaniem układów dynamicznych uwzględniających interakcje zachodzące pomiędzy daną konstrukcją bazową, a zamocowanymi na niej aktywnymi elementami wykonawczymi (aktuatorami).

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