JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
46, 3, pp. 693-714, Warsaw 2008

NONLINEAR NORMAL MODES OF COUPLED SELF-EXCITED OSCILLATORS IN REGULAR AND CHAOTIC VIBRATION REGIMES

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Vibration analysis of coupled self-excited nonlinear oscillators have been studied in the paper. Possible regular motion generated by nonlinear damping has been determined by extracting Nonlinear Normal Modes (NNM) from the model. Influence of the nonlinear terms and intensity of self-excitation on the system response and vibration modes have been presented. Parameters leading to chaotic motion have been found and Nonlinear Normal Modes, which may appear nearby the chaotic response have been presented as well. The considered two degree of freedom example shows that the autonomous system (without time dependent excitation) may transit to chaotic vibrations if the system posseses a potential function with "potential wells". However, NNMs separated for a very sensitive region close to chaotic vibrations, does not converge with motion of the original system in the third approximation order.

Key words: self-excitation, nonlinear vibrations, nonlinear normal modes, chaotic motion

1. Introduction

Self-excited oscillations belong to a special class of vibrations which may occur without any external or internal periodic forcing. They appear due to specific internal properties of a system. As a classical example of a self-excited system we can mention: vibration of airplane wings (flutter), unwanted vibrations during machining processes (chatter), vibrations of a vehicle wheel (shimmy), etc. Two main types of self-excitation are usually distinguished: (a) soft self-excitation represented in the phase space by a stable limit cycle and, (b) hard self-excitation represented by an unstable limit cycle. In the second type, depending on initial conditions, the trajectory tends to an equilibrium point or to infinity. Therefore, this type self-excitation is sometimes called catastrophic (Warmiński, 2001). If the self-excitation interacts with other vibration types, such as an external periodic force or parametric vibrations, many new unexpected phenomena can be observed (Szabelski and Warmiński, 1995, 1997; Warmiński, 2003).

The mechanism of self-excitation can be modelled by differential equations with time delay terms or by introduction of nonlinear functions of the state space coordinates which can model this physical phenomenon. The second approach is discussed in the present paper. The mathematical model of the self-excited system does not include terms directly depending on time. The dynamics is governed by autonomous, ordinary differential equations which include nonlinear terms modelling the self-excitation. Van der Pol's or Rayleigh's models of self-excitation are in common use in literature. They are treated as equivalent, what is not quite true if we consider other nonlinear terms included in the model (Warmiński, 2001). Rayleigh's model of self-excitation, as a more suitable for mechanical systems, is considered in the paper.

Another important problem connected with strongly nonlinear and coupled multi-degree-of-freedom systems concerns a proper extraction of vibration modes. If the system is linear, then its motion can be represented by a superposition of every single vibration mode. When motion of one particle is known, then motion of the rest particles is also determined by a linear functional dependence of the space coordinates. The linear mode separation can be applied to linear and weakly viscously damped systems. The method of NNMs presented by Szemplińska-Stupnicka (1997, 1973) allows for construction of NNM around resonances for externally forced systems.

However, if damping is large or is represented by nonlinear functions, then the response of the system may depend not only on the displacement but also on velocity (Rosenberg, 1960; Vakakis, 1997). In the case of a nonlinear autonomous coupled self-excited system, the method of nonlinear normal modes proposed by Shaw and Pierre (1993) seems to be very promising to its analysis. This method is suitable for strongly nonlinear and also velocity-dependent models. To the author's knowledge, there are no papers in the literature devoted to this topic. Some preliminary results on nonlinear normal modes of self-excited systems were presented by Warmiński (2004, 2006) during two conferences related to NNMs.

The purpose of this paper is to study motion of two coupled, nonlinear, self-excited oscillators in a different parameter configuration by application of nonlinear normal modes, and next to find their most essential properties in the regular and chaotic regimes.

2. Model of the oscillatory system

The considered model is composed of two nonlinear oscillators coupled by a linear spring $f_{12}(x_1, x_2)$. Each oscillator includes nonlinear stiffness, represented by nonlinear functions $f_1(x_1)$ and $f_2(x_2)$, respectively, and nonlinear damping given by $f_{d1}(\dot{x}_1)$, $f_{d2}(\dot{x}_2)$.

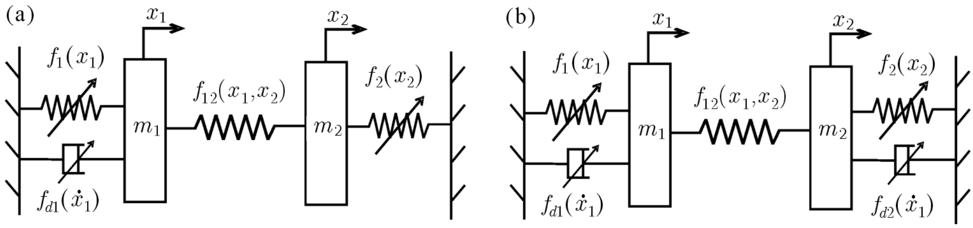

Fig. 1. Physical model of the system with one (a) and two (b) nonlinear dampers

Assuming nonlinear stiffness of Duffing's type and Rayleigh's type of self-excitation, the differential equations of motion take form

$$\begin{aligned} m_1\ddot{x}_1 + (-\alpha_1 + \beta_1\dot{x}_1^2)\dot{x}_1 + \delta_1 x_1 + \gamma_1 x_1^3 + \delta_{12}(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + (-\alpha_2 + \beta_2\dot{x}_2^2)\dot{x}_2 + \delta_2 x_2 + \gamma_2 x_2^3 - \delta_{12}(x_1 - x_2) = 0 \end{aligned} \tag{2.1}$$

where, $f_{d1}(\dot{x}_1) = (-\alpha_1 + \beta_1\dot{x}_1^2)\dot{x}_1$, $f_{d2}(\dot{x}_2) = (-\alpha_2 + \beta_2\dot{x}_2^2)\dot{x}_2$ are nonlinear Rayleigh's functions, and $f_1(x_1) = \delta_1 x_1 + \gamma_1 x_1^3$, $f_2(x_2) = \delta_2 x_2 + \gamma_2 x_2^3$ nonlinear stiffness of Duffing's type. Both oscillators are coupled by a linear spring with the stiffness δ_{12}. Set (2.1) consists of two nonlinear autonomous differential equations without time in an explicit form. Assuming that $\alpha_1 = 0$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = 0$, we get a linear conservative system with natural frequencies determined by the formula

$$\begin{aligned} \omega_{01,2} = \Big\{ \frac{1}{2}(\delta_1 + \delta_{12}) + M(\delta_{12} + \delta_2) \mp \\ \mp\sqrt{[(\delta_1 + \delta_{12}) + M(\delta_{12} + \delta_2)]^2 - 4M(\delta_1\delta_{12} + \delta_1\delta_2 + \delta_{12}\delta_2)} \Big\}^{\frac{1}{2}} \end{aligned} \tag{2.2}$$

where $M = m_1/m_2$.

Nevertheless, due to existing nonlinear terms which either depend on the displacement or velocity, the decoupling of the system should take into account both aspects: nonlinear stiffness and nonlinear damping. Therefore, the nonlinear normal modes will be constructed for a strongly nonlinear system.

3. Non-linear normal modes

For the nonlinear normal modes formulation the set of equations (2.1) is rewritten in the form

$$\begin{aligned} \dot{x}_1 &= y_1 & \dot{y}_1 &= f_1(x_1, y_1, x_2, y_2) \\ \dot{x}_2 &= y_2 & \dot{y}_2 &= f_2(x_1, y_1, x_2, y_2) \end{aligned} \tag{3.1}$$

where

$$f_1 \equiv f_1(x_1, y_1, x_2, y_2) = \frac{1}{m_1}\{-\delta_1 x_1 - \delta_{12}(x_1 - x_2) + \varepsilon[(\widetilde{\alpha}_1 - \widetilde{\beta}_1 y_1^2)y_1 - \widetilde{\gamma}_1 x_1^3]\}$$

$$f_2 \equiv f_2(x_1, y_1, x_2, y_2) = \frac{1}{m_2}\{-\delta_2 x_2 + \delta_{12}(x_1 - x_2) + \varepsilon[(\widetilde{\alpha}_2 - \widetilde{\beta}_2 y_2^2)y_2 - \widetilde{\gamma}_2 x_2^3]\}$$

The coefficients of nonlinear terms are expressed as

$$\begin{aligned} \alpha_1 &= \varepsilon\widetilde{\alpha}_1 & \beta_1 &= \varepsilon\widetilde{\beta}_1 \\ \alpha_2 &= \varepsilon\widetilde{\alpha}_2 & \beta_2 &= \varepsilon\widetilde{\beta}_2 \end{aligned}$$

where $\varepsilon > 0$ denotes a formal small parameter which allows for grouping of all nonlinear terms.

The coordinates x_1 and y_1 of the first oscillator are chosen as master coordinates, and they are denoted as

$$x_1 = u \qquad y_1 = v \tag{3.2}$$

According to Shaw and Pierre (1993), the coordinates of the second oscillator (slave coordinates) are expressed as functions of the master coordinates

$$x_2 = X_2(u, v) \qquad y_2 = Y_2(u, v) \tag{3.3}$$

We assume that the functional dependencies $X_2(u, v)$ and $Y_2(u, v)$ exist, thus the displacement and velocity of the second oscillator can be expressed by the displacement and velocity of the first oscillator. The functions $X_2(u, v)$,

$Y_2(u,v)$ are constraint equations and they represent, the so called, modal surfaces. Their time derivatives can be found as

$$\dot{X}_2(u,v) = \frac{\partial X_2}{\partial u}\dot{u} + \frac{\partial X_2}{\partial v}\dot{v} \qquad \dot{Y}_2(u,v) = \frac{\partial Y_2}{\partial u}\dot{u} + \frac{\partial Y_2}{\partial v}\dot{v} \tag{3.4}$$

Expressing differential equations of motion (3.1) by the master-slave coordinate notation yields

$$\begin{aligned} &v = \dot{u} && \dot{v} = f_1(u,v,X_2(u,v),Y_2(u,v)) \\ &\dot{x}_2 \equiv Y_2(u,v) && \dot{y}_2 \equiv \dot{Y}_2(u,v) = f_2(u,v,X_2(u,v),Y_2(u,v)) \end{aligned} \tag{3.5}$$

and next, substituting the above equations into (3.4), we get

$$\begin{aligned} Y_2(u,v) &= \frac{\partial X_2(u,v)}{\partial u}v + \frac{\partial X_2(u,v)}{\partial v}f_1(u,v,X_2(u,v),Y_2(u,v)) \\ f_2(u,v,X_2(u,v),Y_2(u,v)) &= \frac{\partial Y_2(u,v)}{\partial u}v + \frac{\partial Y_2(u,v)}{\partial v}f_1(u,v,X_2(u,v),Y_2(u,v)) \end{aligned} \tag{3.6}$$

Equations (3.6) determine the modal surfaces. However, bearing in mind that the considered model is nonlinear, receiving analytical solutions in the general case can produce difficulties or can be even impossible. Therefore, assuming that the oscillations take place around the zero equilibrium position, we can expand the constraint functions in power series

$$\begin{aligned} X_2(u,v) &= a_1u + a_2v + a_3u^2 + a_4uv + a_5v^2 + a_6u^3 + a_7u^2v + \\ &\quad + a_8uv^2 + a_9v^3 + \ldots \\ Y_2(u,v) &= b_1u + b_2v + b_3u^2 + b_4uv + b_5v^2 + b_6u^3 + b_7u^2v + \\ &\quad + b_8uv^2 + b_9v^3 + \ldots \end{aligned} \tag{3.7}$$

The expansion takes into account terms up to the cubic order, which is in agreement with nonlinear functions included in Eq. (2.1). Time derivatives of Eqs. (3.7) take forms

$$\begin{aligned} \dot{x}_2 = \dot{X}_2(u,v) &= a_1\dot{u} + a_2\dot{v} + 2a_3u\dot{u} + a_4u\dot{v} + a_4\dot{u}v + 2a_5v\dot{v} + 3a_6u^2\dot{u} + \\ &\quad + 2a_7u\dot{u}v + a_7u^2\dot{v} + a_8\dot{u}v^2 + 2a_8uv\dot{v} + 3a_9v^2\dot{v} + \ldots \\ \dot{y}_2 = \dot{Y}_2(u,v) &= b_1\dot{u} + b_2\dot{v} + 2b_3u\dot{u} + b_4u\dot{v} + b_4\dot{u}v + 2b_5v\dot{v} + 3b_6u^2\dot{u} \\ &\quad + 2b_7u\dot{u}v + b_7u^2\dot{v} + b_8\dot{u}v^2 + 2b_8uv\dot{v} + 3b_9v^2\dot{v} + \ldots \end{aligned} \tag{3.8}$$

Considering the above expressions, the last two equations of (3.1) and going back to the original notation, we get

$$
\begin{aligned}
y_2 &= a_1y_1 + a_2f_1 + 2a_3x_1y_1 + a_4x_1f_1 + a_4y_1^2 + 2a_5y_1f_1 + 3a_6x_1^2y_1 + 2a_7x_1y_1^2 \\
&\quad + a_7x_1^2f_1 + a_8y_1^3 + 2a_8x_1y_1f_1 + 3a_9y_1^2f_1 + \ldots \\
f_2 &= b_1y_1 + b_2f_1 + 2b_3x_1y_1 + b_4x_1f_1 + b_4y_1^2 + 2b_5y_1f_1 + 3b_6x_1^2y_1 + \\
&\quad + 2b_7x_1y_1^2 + b_7x_1^2f_1 + b_8y_1^3 + 2b_8x_1y_1f_1 + 3b_9y_1^2f_1 + \ldots
\end{aligned}
\tag{3.9}
$$

Next we substitute (3.7) and (3.8) into functions $f_1(x_1,y_1,x_2,y_2) = f_1(u,v,X_2(u,v),Y_2(u,v))$, $f_2(x_1,y_1,x_2,y_2) = f_2(u,v,X_2(u,v),Y_2(u,v))$ in Eqs. (3.9).

From the last two equations of (3.1), it results that

$$
y_2 - \dot{x}_2 = 0 \qquad\qquad f_2(x_1,y_1,x_2,y_2) - \dot{y}_2 = 0 \tag{3.10}
$$

Thus, grouping the terms of (3.10) in a proper order with respect to the master coordinates, we receive a set of two equations composed of the terms: u, v, u^2, uv, v^2, u^3, u^2v, uv^2, v^3, $\ldots$, v^9. Equation (3.10) is satisfied for non-trivial u and v, only if the coefficients near the mentioned terms are equal to zero. It allows one to determine the unknown parameters $a_1, a_2, \ldots$, $b_1, b_2, \ldots$ of expansion (3.7). Taking into account that the highest order terms are negligible, it has been decided to solve the problem up to the third order of the coordinates u and v. Terms of higher order are truncated from the expansion. Eventually, we get a set of eighteenth algebraic nonlinear equations (nine for each of Eqs. (3.10)) with eighteen unknown parameters $a_1, \ldots, a_9$, $b_1, \ldots, b_9$.

The parameters of modal surfaces are determined from equations:

— term u

$$
\begin{aligned}
&b_1 + \frac{a_2}{m_1}(\delta_1 + \delta_{12} + a_1\delta_{12}) = 0 \\
&\frac{b_2}{m_1}(\delta_1 + \delta_{12} - a_1\delta_{12}) + \frac{1}{m_2}[\delta_{12} - a_1(\delta_{12} - \delta_2) + \varepsilon b_1\widetilde{\alpha}_2] = 0
\end{aligned}
\tag{3.11}
$$

— term v

$$
\begin{aligned}
&-a_1 + b_2 - \frac{a_2}{m_1}(a_2\delta_{12} + \varepsilon\widetilde{\alpha}_1) = 0 \\
&-b_1 + \frac{1}{m_1}(-a_2b_2\delta_{12} - b_2\varepsilon\widetilde{\alpha}_1) + \frac{1}{m_2}[a_2(-\delta_{12} - \delta_2) + b_2\varepsilon\widetilde{\alpha}_2] = 0
\end{aligned}
\tag{3.12}
$$

— term u^2

$$\begin{aligned}
&b_3+\frac{1}{m_1}[-a_2a_3\delta_{12}-a_1a_4\delta_{12}+a_4(\delta_1+\delta_{12})]=0\\
&\frac{1}{m_1}[-a_3b_2\delta_{12}-a_1b_4\delta_{12}+b_4(\delta_1+\delta_{12})]+\frac{1}{m_2}[a_3(-\delta_{12}-\delta_2)+b_3\varepsilon\widetilde{\alpha}_2]=0
\end{aligned}\tag{3.13}$$

— term uv

$$\begin{aligned}
&-2a_3+b_4+\frac{1}{m_1}[-2a_2a_4\delta_{12}-2a_1a_5\delta_{12}+a_5(2\delta_1+2\delta_{12})-a_4\varepsilon\widetilde{\alpha}_1]=0\\
&-2b_3+\frac{1}{m_1}[-a_4b_2\delta_{12}-a_2b_4\delta_{12}-2a_1b_5\delta_{12}+b_5(2\delta_1+2\delta_{12})-b_4\varepsilon\widetilde{\alpha}_1]+\\
&\qquad+\frac{1}{m_2}[a_4(-\delta_{12}-\delta_2)+b_4\varepsilon\widetilde{\alpha}_2]=0
\end{aligned}\tag{3.14}$$

— term v^2

$$\begin{aligned}
&-a_4+b_5+\frac{1}{m_1}(-3a_2a_5\delta_{12}-2a_5\varepsilon\widetilde{\alpha}_1)=0\\
&-b_4+\frac{1}{m_1}(-a_5b_2\delta_{12}-2a_2b_5\delta_{12}-2b_5\varepsilon\widetilde{\alpha}_1)+\frac{1}{m_2}[a_5(-\delta_{12}-\delta_2)+b_5\varepsilon\widetilde{\alpha}_2]=0
\end{aligned}\tag{3.15}$$

— term u^3

$$\begin{aligned}
&b_6+\frac{1}{m_1}[-a_3a_4\delta_{12}-a_1a_7\delta_{12}+a_7(\delta_1+\delta_{12})+a_2(-a_6\delta_{12}+\varepsilon\widetilde{\gamma}_1)]=0\\
&\frac{1}{m_1}[-a_6b_2\delta_{12}-a_3b_4\delta_{12}-a_1b_7\delta_{12}+b_7(\delta_1+\delta_{12})+b_2\varepsilon\widetilde{\gamma}_1]+\\
&\qquad+\frac{1}{m_2}[a_6(-\delta_{12}-\delta_2)+b_6\varepsilon\widetilde{\alpha}_2-b_1^3\varepsilon\widetilde{\beta}_2-a_1^3\varepsilon\widetilde{\gamma}_2]=0
\end{aligned}\tag{3.16}$$

— term u^2v

$$\begin{aligned}
&-3a_6+b_7+\frac{1}{m_1}[-a_4^2\delta_{12}-2a_3a_5\delta_{12}-2a_2a_7\delta_{12}-2a_1a_8\delta_{12}+\\
&\qquad+a_8(2\delta_1+2\delta_{12})-a_7\varepsilon\widetilde{\alpha}_1]=0\\
&-3b_6+\frac{1}{m_1}[-a_7b_2\delta_{12}-a_4b_4\delta_{12}-2a_3b_5\delta_{12}-a_2b_7\delta_{12}-2a_1b_8\delta_{12}+\\
&\qquad+b_8(2\delta_1+2\delta_{12})-b_7\varepsilon\widetilde{\alpha}_1]+\\
&\qquad+\frac{1}{m_2}[a_7(-\delta_{12}-\delta_2)+b_7\varepsilon\widetilde{\alpha}_2-3b_1^2b_2\varepsilon\widetilde{\beta}_2-3a_1^2a_2\varepsilon\widetilde{\gamma}_2]=0
\end{aligned}\tag{3.17}$$

— term uv^2

$$
\begin{gathered}
-2a_7 + b_8 + \\
+\frac{1}{m_1}[-3a_4a_5\delta_{12} - 3a_2a_8\delta_{12} - 3a_1a_9\delta_{12} + a_9(3\delta_1 + 3\delta_{12}) - 2a_8\varepsilon\alpha_1] = 0 \\
-2b_7 + \frac{1}{m_1}[-a_8b_2\delta_{12} - a_5b_4\delta_{12} - 2a_4b_5\delta_{12} - 2a_2b_8\delta_{12} - 3a_1b_9\delta_{12} + \\
+b_9(3\delta_1 + 3\delta_{12}) - 2b_8\varepsilon\widetilde{\alpha}_1] + \\
+\frac{1}{m_2}[a_8(-\delta_{12} - \delta_2) + b_8\varepsilon\widetilde{\alpha}_2 - 3b_1b_2^2\varepsilon\widetilde{\beta}_2 - 3a_1a_2^2\varepsilon\widetilde{\gamma}_2] = 0
\end{gathered}
\tag{3.18}
$$

— term v^3

$$
\begin{gathered}
-a_8 + b_9 + \frac{1}{m_1}[-2a_5^2\delta_{12} - 3a_9\varepsilon\widetilde{\alpha}_1 + a_2(-4a_9\delta_{12} + \varepsilon\widetilde{\beta}_1)] = 0 \\
-b_8 + \frac{1}{m_1}[-a_9b_2\delta_{12} - 2a_5b_5\delta_{12} - 3a_2b_9\delta_{12} - 3b_9\varepsilon\widetilde{\alpha}_1 + b_2\varepsilon\widetilde{\beta}_1] + \\
+\frac{1}{m_2}[a_9(-\delta_{12} - \delta_2) + b_9\varepsilon\widetilde{\alpha}_2 - b_2^3\varepsilon\widetilde{\beta}_2 - a_2^3\varepsilon\widetilde{\gamma}_2] = 0
\end{gathered}
\tag{3.19}
$$

The parameters a_1, a_2, b_1, b_2 can be found independently of the first four equations, while a_3, a_4, a_5, b_3, b_4, b_5 are equal to zero because of lack of quadratic nonlinear terms in the original dynamic equations, the rest, i.e. a_6, a_7, a_8, a_9, b_6, b_7, b_8, b_9 are found for the assumed numerical data.

As can be noticed in the next Section, in the numerical example, the set of equations (3.11)-(3.19) gives two different real solutions which represent the first and second nonlinear normal vibration modes. Substituting these solutions into the first two equations (3.1), we obtain uncoupled nonlinear differential equations for the first and second mode, respectively

$$
v_i = \dot{u}_i \qquad \dot{v}_i = f_{1i}(u_i, v_i, X_{2i}(u_i, v_i), Y_{2i}(u_i, v_i)) \qquad i = 1, 2
\tag{3.20}
$$

Taking into account that $\dot{v}_i = \ddot{u}_i$, equations (3.20) take form:

— for the first mode

$$
\ddot{u}_1 + \omega_{01}^2 u_1 + (-\alpha_I + \beta_I \dot{u}^2)\dot{u}_1 + (\eta_I u_1 + \rho_I \dot{u}_1)u_1\dot{u}_1 + \gamma_I u_1^3 = 0
\tag{3.21}
$$

— for the second mode

$$
\ddot{u}_2 + \omega_{02}^2 u_2 + (-\alpha_{II} + \beta_{II} \dot{u}_2^2)\dot{u}_2 + (\eta_{II} u_2 + \rho_{II} \dot{u}_2)u_2\dot{u}_2 + \gamma_{II} u_2^3 = 0
\tag{3.22}
$$

The coefficients ω_{01}, ω_{02} represent natural frequencies of the linear system and they are in full accordance with linear eigenvalues (2.2). The coefficients α_I, α_{II}, β_I, β_{II} are called modal coefficients of Rayleigh's self-excitation, γ_I, γ_{II}, η_I, η_{II}, ρ_I, ρ_{II} – modal nonlinear terms of the first and second vibration modes, respectively.

4. Numerical example of regular motion

Exemplary calculations have been done for following data (Warmiński, 2001):
— variant I – one self-excited damper

$$\begin{array}{llll} \alpha_1 = 0.01 & \alpha_2 = 0 & \beta_1 = 0.05 & \beta_2 = 0 \\ \gamma_1 = 0.1 & \gamma_2 = 0.1 & \delta_1 = \delta_2 = 1 & \delta_{12} = 0.3 \\ m_1 = 1 & m_2 = 2 & & \end{array} \tag{4.1}$$

— variant II – two self-excited dampers

$$\begin{array}{lll} \alpha_1 = \alpha_2 = 0.01 & \beta_1 = \beta_2 = 0.05 & \\ \delta_1 = \delta_2 = 1 & \delta_{12} = 0.3 & \\ \gamma_1 = \gamma_2 = 0.1 & m_1 = 1 & m_2 = 2 \end{array} \tag{4.2}$$

Natural frequencies calculated from (2.2) take values: $\omega_{01} = 0.766$, $\omega_{02} = 1.168$, while the coefficients necessary for separation of the nonlinear modes $a_1, \dots, a_9$, $b_1, \dots, b_9$ defined in the previous section take values (Table 1).

After application of the decoupling procedure, we get a differential equation for the two non-linear oscillators:
— variant I – one self-excited damper
(a) mode I

$$\begin{aligned} &\ddot{u} + 0.5869u + (-0.000813 + 0.0163045\dot{u}^2)\dot{u} + \\ &+(-0.02387u + 0.25684\dot{u})u\dot{u} + 0.2174u^3 = 0 \end{aligned} \tag{4.3}$$

(b) mode II

$$\begin{aligned} &\ddot{u} + 1.3631u + (-0.01 + 0.04669\dot{u}^2)\dot{u} + \\ &-(0.00414u + 0.00384\dot{u})u\dot{u} + 0.09418u^3 = 0 \end{aligned} \tag{4.4}$$

Table 1. Coefficients of nonlinear normal modes

Variant I				
Mode I				
$a_1 = 2.377$	$a_2 = -0.03062$	$a_3 = 0$	$a_4 = 0$	$a_5 = 0$
$a_6 = -0.3926$	$a_7 = 0.0796$	$a_8 = -0.8559$	$a_9 = 0.112318$	
$b_1 = 0.017973$	$b_2 = 2.37693$	$b_3 = 0$	$b_4 = 0$	$b_5 = 0$
$b_6 = -0.04004$	$b_7 = -0.1697$	$b_8 = -0.03216$	$b_9 = -0.855365$	
Mode II				
$a_1 = -0.2103$	$a_2 = -0.00271$	$a_3 = 0$	$a_4 = 0$	$a_5 = 0$
$a_6 = 0.0194$	$a_7 = 0.0138$	$a_8 = 0.013$	$a_9 = 0.011$	
$b_1 = 0.00369$	$b_2 = -0.21033$	$b_3 = 0$	$b_4 = 0$	$b_5 = 0$
$b_6 = -0.01854$	$b_7 = 0.0234$	$b_8 = -0.0174$	$b_9 = 0.0132$	
Variant II				
Mode I				
$a_1 = 2.377$	$a_2 = -0.01531$	$a_3 = 0$	$a_4 = 0$	$a_5 = 0$
$a_6 = -0.3926$	$a_7 = -0.1376$	$a_8 = -0.8559$	$a_9 = -0.1843$	
$b_1 = 0.00899$	$b_2 = 2.37693$	$b_3 = 0$	$b_4 = 0$	$b_5 = 0$
$b_6 = 0.0841$	$b_7 = -0.1734$	$b_8 = 0.044$	$b_9 = -0.8572$	
Mode II				
$a_1 = -0.2103$	$a_2 = -0.001354$	$a_3 = 0$	$a_4 = 0$	$a_5 = 0$
$a_6 = 0.0196$	$a_7 = 0.0133$	$a_8 = 0.013$	$a_9 = 0.011$	
$b_1 = 0.00185$	$b_2 = -0.21035$	$b_3 = 0$	$b_4 = 0$	$b_5 = 0$
$b_6 = -0.018$	$b_7 = 0.0236$	$b_8 = -0.0169$	$b_9 = 0.0134$	

— variant II – two self-excited dampers

(a) mode I

$$\begin{aligned}&\ddot{u} + 0.5869u + (-0.00541 + 0.10531\dot{u}^2)\dot{u} +\\ &+(0.04129u + 0.25676\dot{u})u\dot{u} + 0.2178u^3 = 0\end{aligned} \tag{4.5}$$

(b) mode II

$$\begin{aligned}&\ddot{u} + 1.3631u + (-0.01 + 0.0468\dot{u}^2)\dot{u} -\\ &-(0.00398u + 0.00394\dot{u})u\dot{u} + 0.094113u^3 = 0\end{aligned} \tag{4.6}$$

Equations (4.3), (4.4) and (4.5), (4.6) represent uncoupled motion of two nonlinear oscillators for each nonlinear mode and for the first and second variant, respectively. Note that the coefficients of the linear components u obtained after transformation are equal to the square of the natural frequency determined in a classical way, (2.2), which confirms correctness of the transformation.

Comparing both numerical variants, we see that the equations have similar nature (posses the same nonlinear terms), only values of the coefficients are different.

To verify the analytical results, equations (4.3), (4.4) and adequately (4.5), (4.6) are solved numerically and the results are compared with a direct numerical simulation of original equations (3.1).

Because the system is self-excited, there is no special leading excitation frequency related with natural frequencies of the system. Numerical analysis shows that depending on initial conditions, the first or the second mode of the system can be activated. On the phase plane x_1y_1 (Fig. 2a) we get two limit cycles which represent the first and the second nonlinear mode. Corresponding basins of attraction (BA) of two possible solutions are presented in the phase plane x_2y_2 in Fig. 2b.

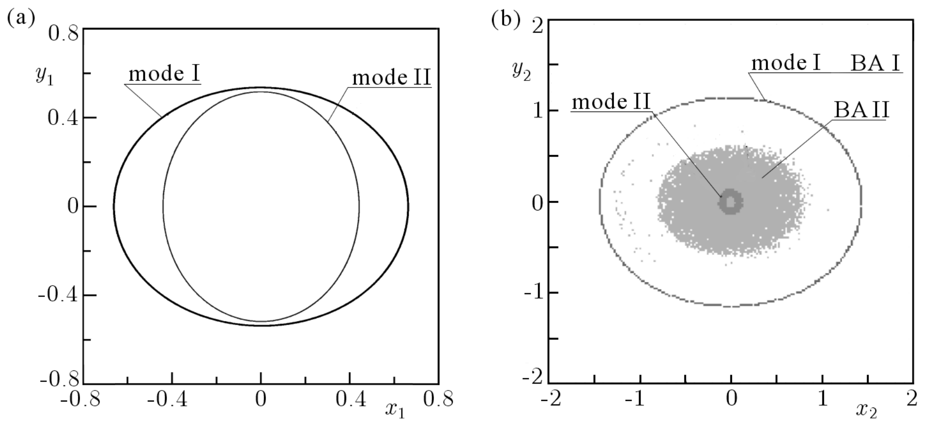

Fig. 2. Phase plane x_1y_1 with limit cycles (a) and basins of attraction in the phase plane x_2y_2 (b) for the first and second vibration modes; model with one self-excitation term (variant I, (4.1))

The time histories in Fig. 3 represent motion of the system in physical and normal coordinates. The normal coordinates have been obtained by numerical integration and the inverse transformation of the physical coordinates (x_1, y_1, x_2, y_2) in the coordinates (u_1, v_1, u_2, v_2). In Fig. 3a, the first vibration mode is activated – evidently synchronous motion of both masses takes place. After transformation to nonlinear normal modes, by using (4.3) and (4.4), we get motion expressed only by the first normal mode u_1, while the second mode u_2 is close to zero (Fig. 3b). The second mode visible in Fig. 3c corresponds to anti-symmetric motion. After transformation to the normal coordinates

(Fig. 3d), the motion is represented only by the second normal mode u_2. The first mode u_1 is not activated.

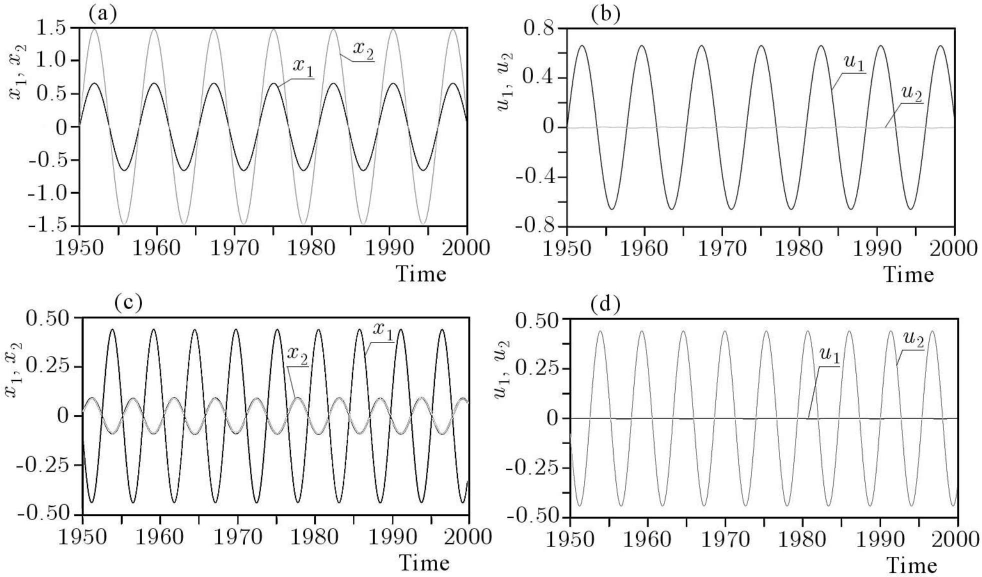

Fig. 3. Time histories in physical and normal coordinates obtained for the first (a) and the second (b) vibration mode; model with one self-excitation term (variant I, (4.1))

After application of the decoupling procedure, the vibration modes are represented by two independent oscillators described by (4.3) and (4.4). Those equations include nonlinear damping and nonlinear stiffness which play important role in the system dynamics. This is the most essential difference between the classical linear mode transformation and that presented in this paper. Motion of each nonlinear oscillator is represented by a limit cycle, which means that NNMs fit well qualitatively to the original self-excited model.

In the second variant, (4.2), two self-excited dampers are included (Fig. 1b). We can expect that, in the general case, two modes of the system can be activated. Of course, for a specific set of parameters, also one-mode response is possible, or when the system satisfies some symmetry conditions, so called similar normal modes can be obtained. Then motion of the nonlinear system is manifested in modes similar to its linear counterpart. Analysis of these specific features are out of scope of this paper. In general, the system with two self-excitations is composed of two nonlinear normal modes. Time histories of physical coordinates obtained for such a case (data (4.2)) are presented in Fig.4a and 4b.

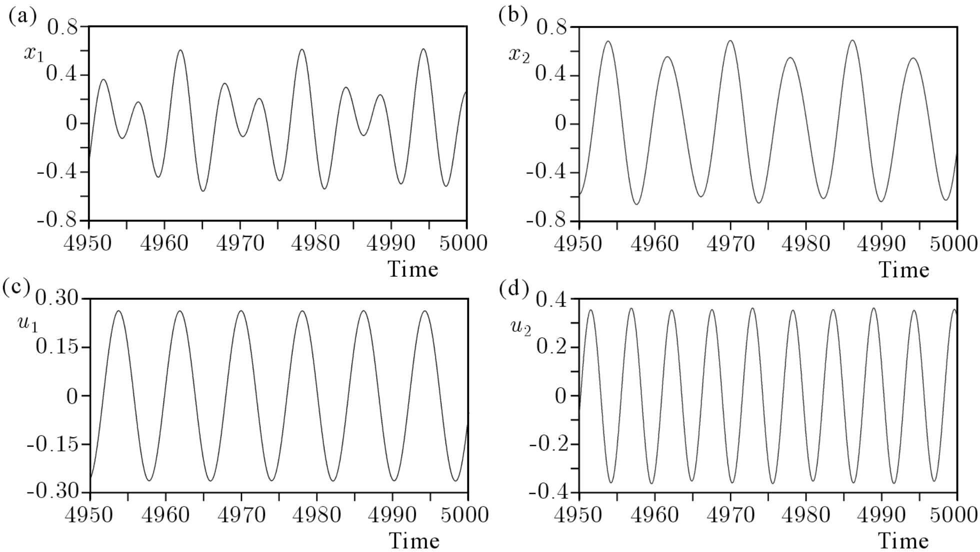

Fig. 4. Time histories in physical coordinates (a), (b) and normal coordinates obtained for the first (c) and the second (d) vibration mode; model composed of two self-excitation terms (variant II, (4.2))

After nonlinear transformation, by using the second set of coefficients (Table 1) we get a modal response of the system. Two extracted normal modes u_1 and u_2 are presented in Fig. 4c and 4d. The motion is very well separated.

The slave coordinates x_2, y_2 are related with the master ones u, v by the modal functions $X_2(u,v)$, $Y_2(u,v)$. These modal surfaces for the first and second modes, respectively, are presented in Fig. 5.

An exemplary trajectory which tends to a stable limit cycle and stays on the modal surfaces is also plotted in this figure. Decoupling of motion on the two self-excited oscillators is clearly visible. As results from the above figures, the nonlinear modal surfaces strongly depend both on the displacement and velocity. It strictly comes from the self-excitation terms which are nonlinear functions of velocity. In the classical linear approach, the influence of velocity is neglected.

To activate only one mode, we start from the initial conditions $u_{t=0} = 1$ and $v_{t=0} = 0$ which, after functional transformation, leads to the following initial conditions of the original system: $x_{10} = 1$, $y_{10} = 0$, $x_{20} = 1.98437$, $y_{20} = 0.0931$ for mode I, and $x_{10} = 1$, $y_{10} = 0$, $x_{20} = -0.190714$, $y_{20} = -0.01611$ for mode II.

A trial of activation only one-mode vibrations by putting a proper initial condition does not lead to a single mode response of the system. At the

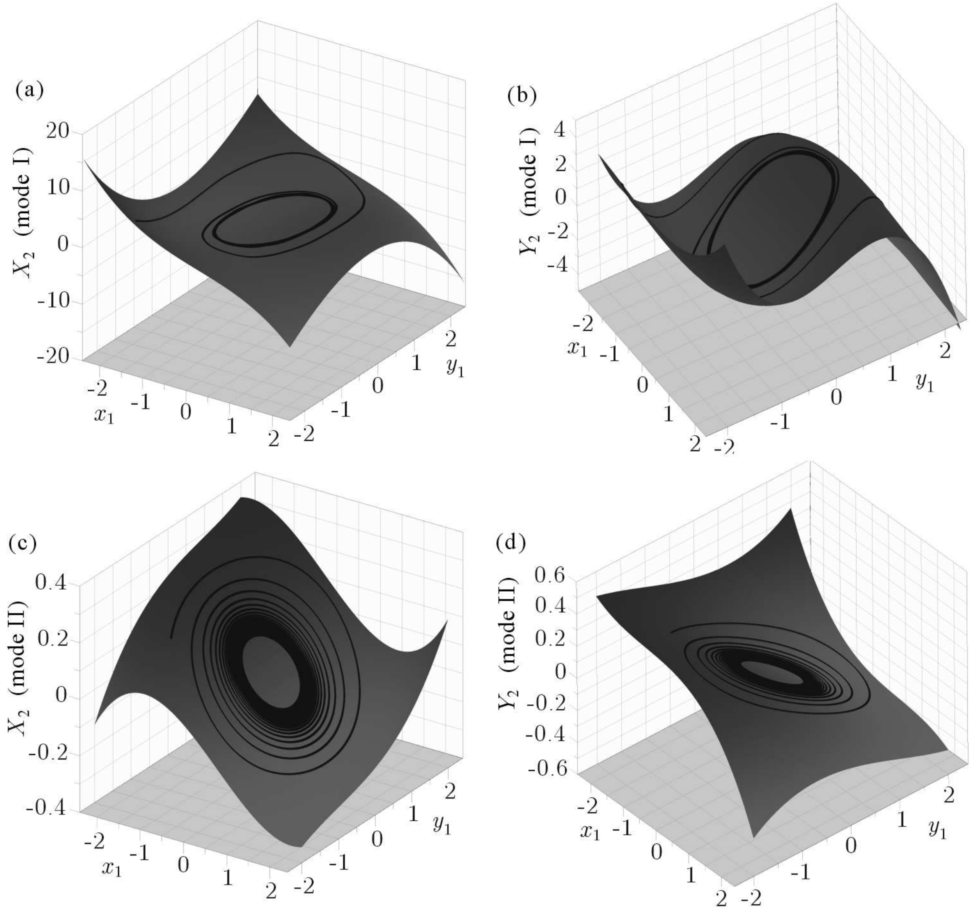

Fig. 5. Nonlinear modal surfaces; model with two self-excitations terms – variant II; (a), (b) mode I, (c), (d) mode II

beginning, only one mode dominates (Fig. 6), but due to a small numerical perturbation, the second mode, in long time period, is also activated (Fig. 7). This is caused by the soft type of self-excitation, which means that the equilibrium point is unstable and the trajectories tend to a stable quasi-periodic limit cycle represented by a torus in the phase space.

"Superposition" of the solutions obtained from both separated oscillators should correspond to the result obtained from direct numerical simulation of the original system.

In Fig. 8a, we see a projection of a quasi-periodic torus on the phase plane of the master coordinates, obtained from the direct numerical simulation of

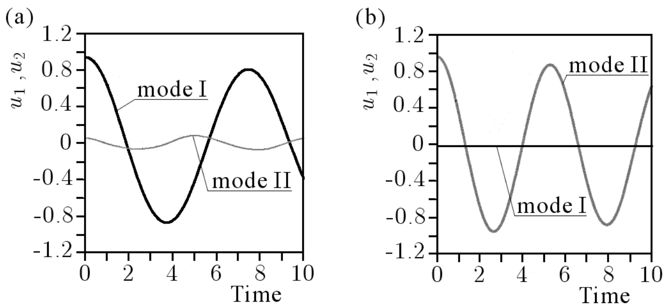

Fig. 6. Time histories in normal coordinates obtained for the first (a) and the second (b) vibration mode; model with two self-excitation terms (variant II, (4.2)), short time after activation

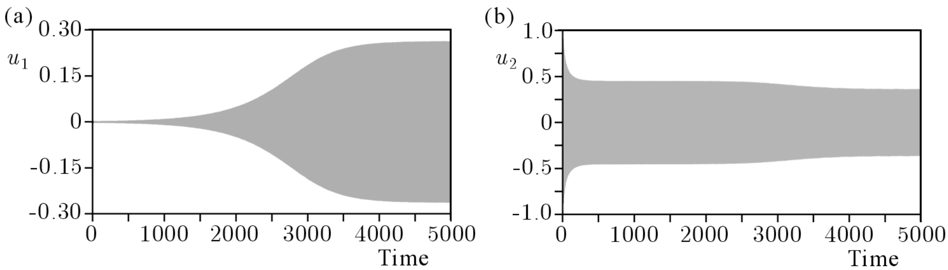

Fig. 7. Time histories in normal coordinates obtained for the first (a) and the second (b) vibration modes, model with two self-excitation terms (variant II, (4.2)), long time after activation

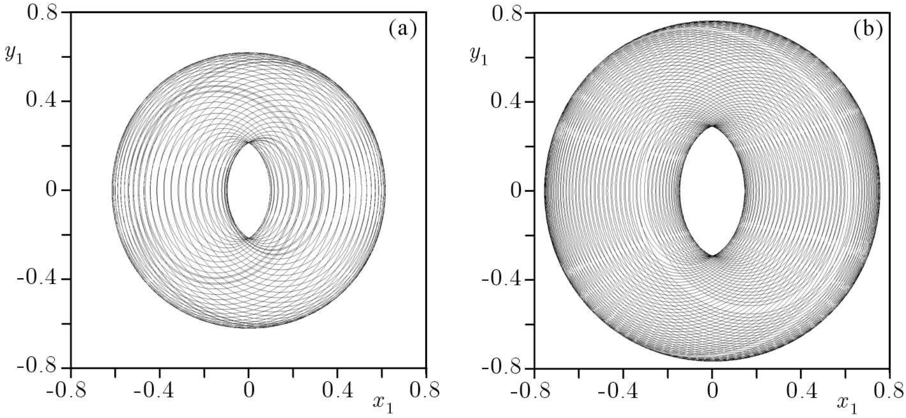

Fig. 8. Quasi periodic tori projected on the physical master coordinates x_1, y_1 obtained from the original system (a), and after superposition of motion of modal oscillators (b); model with two self-excitation terms (variant II, (4.2))

original equations (3.1), and in Fig. 8b superposition of modal oscillators (4.5), (4.6). Both results are qualitatively in a good agreement, but a quantitative difference is visible. The difference comes from the approximation of nonlinear functions of equations (3.9) by truncation terms higher than the third order. For a strongly nonlinear model and large values of the displacement and velocity, it leads to a quantitative difference.

The proposed technique by application of NNM allows for separation of motion. It takes into account the influence of nonlinear terms with respect to the displacement and velocity. However, the basic assumption of this approach is that there exists a functional relation between the master and slave coordinates. This condition can not be always satisfied. In the next Section, we present more complicated dynamics of the considered system.

5. Chaotic response of the model

On the basis of the literature study, we can expect that a two degree of freedom system, even without time-dependent excitation, may transit to chaotic motion. In Warmiński (2001), the transition from regular to chaotic motion was found for a system possessing a four-well potential function.

Let us consider a simpler variant of the model, i.e. variant I with one self-excited damper. To get the required potential function, we assume a strongly nonlinear system composed of two nonlinear oscillators having a negative linear part of the stiffness and large nonlinear terms. Numerical analysis of the system is performed for the following data

$$\begin{array}{llll} \alpha_1 = 0.01 & \alpha_2 = 0 & \beta_1 = 0.05 & \beta_2 = 0 \\ \gamma_1 = 3 & \gamma_2 = 3 & \delta_1 = \delta_2 = -1 & \delta_{12} = 0.3 \\ m_1 = 1 & m_2 = 2 & & \end{array} \tag{5.1}$$

Note that δ_1, δ_2 have negative values and γ_1, γ_2 are large. The potential surface is determined by a function

$$V = \frac{1}{2}\delta_1(x_1^2 + x_2^2) + \frac{1}{2}\delta_{12}(x_1 - x_2)^2 + \frac{1}{4}(\gamma_1 x_1^4 + \gamma_2 x_2^4) \tag{5.2}$$

The potential $V(x_1, x_2)$ for the assumed data is plotted in Fig. 9. This function has four local minima ("wells") and one maximum at the origin of the coordinate system. A cross-section of the surface for $x_2 = 0$ is presented in Fig. 9b.

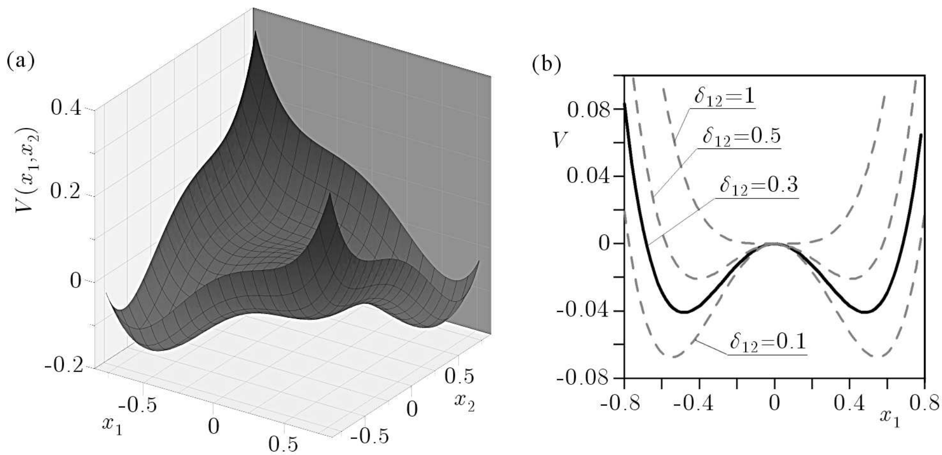

Fig. 9. Potential surface versus system physical coordinates (a) and the cross-section for $x_2 = 0$ (b)

Coupling of both oscillators depends on the connecting spring stiffness, which is denoted by the parameter δ_{12}. Therefore, the potential surface cross-sections are plotted for a few different coupling parameters (dashed lines in Fig. 9). The solid line denotes the function plotted for the main value of this parameter $\delta_{12} = 0.3$. We see that for $\delta_{12} = 1$ the potential wells vanish.

To check the influence of this stiffness and its possible behaviour, the Lyapunov exponents versus the bifurcation parameter δ_{12} are computed (Fig. 10). In the range of the bifurcation parameter $\delta_{12} \in (\sim 0.12, \sim 0.5)$, the maximal Lyapunov exponent is positive, which means that the response of the system is chaotic. Poincaré maps for the parameter $\delta_{12} = 0.3$ are presented in Fig. 11.

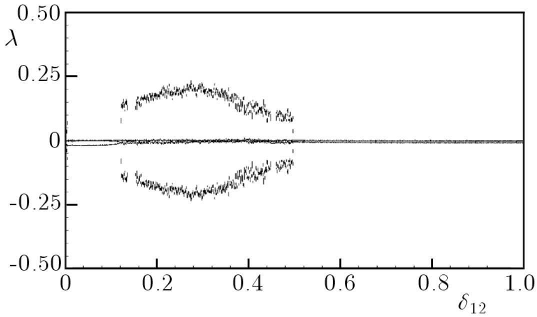

Fig. 10. Lyapunov exponents versus bifurcation parameter δ_{12}

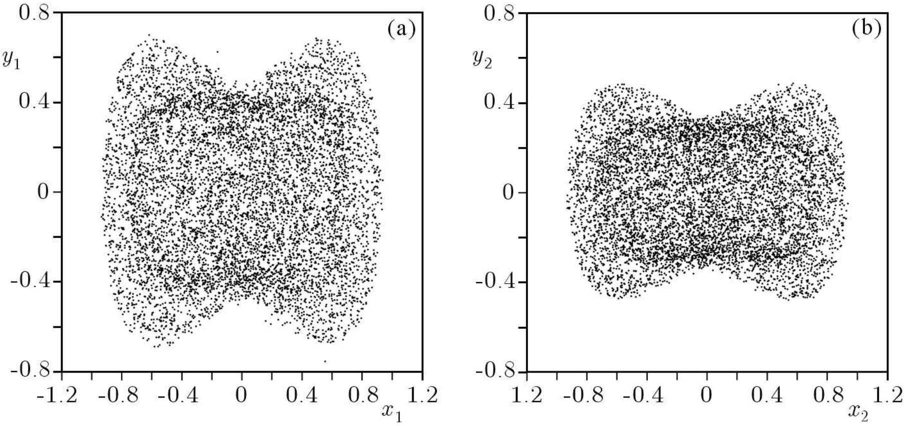

Fig. 11. Strange chaotic attractor in x_1y_1 (a), and x_2y_2 (b) Poincaré sections; model with one self-excitation term (variant (5.1))

Motion of the four-well potential system for assumed data (5.1) becomes irregular, and its nature is represented by a strange chaotic attractor (Fig. 11). If we apply the decoupling strategy by introducing NNM, we may expect to get two nonlinear modal oscillators which should give a chaotic response of the system. These modal oscillators would be one-degree-of-freedom and autonomous subsystems. Because uncoupled modal oscillators belong to 2D phase space, therefore these systems can not produce chaotic motion. At least a 3D phase space model is required to perform chaotic motion. It allows us to conclude that it is not possible to receive a chaotic response as a composition ("superposition") of two 2D nonlinear subsystems.

However, the methodology presented in Section 4 can be formally applied for parameters assumed in (5.1). It means that out of the chaotic region, motion of the slave coordinates x_2, y_2 can be related with the master ones x_1, y_1.

Repeating the procedure presented in details in Section 4, we receive for $\delta_{12} = 0.6$ (which corresponds to regular motion, see Fig. 10), two modal oscillators:

— mode I

$$\ddot{u} - 0.7359u + (-0.00615 + 0.00866\dot{u}^2)\dot{u} + (0.0284u + 0.8661\dot{u})u\dot{u} + 2.23714u^3 = 0 \tag{5.3}$$

— mode II

$$\ddot{u} - 0.1359u + (-0.00385 - 0.0459\dot{u}^2)\dot{u} - (0.01915u + 3.395\dot{u})u\dot{u} + 2.04518u^3 = 0 \tag{5.4}$$

Both oscillators posses a negative linear part of the stiffness, which comes from the assumed parameters. Direct numerical simulations shows that for $\delta_{12} = 0.6$ regular motion takes place and, depending on initial conditions, three different attractors are obtained. These attractors, numbered by 1, 2, 3, are presented in Fig. 12 on Poincaré sections with respect to x_1, y_1 and x_2, y_2 coordinates. The attractors show possible motion around a single potential well (attractors 2 and 3) and they are shifted with respect to the equilibrium point, or motion around two potential wells (attractor 1) with full symmetry about the equilibrium position. It is worth to add that the trajectory reaches the attractor after a long time of irregular motion, which represents a kind of temporary chaos. This means that the system is very sensitive to initial conditions and small perturbations of parameters.

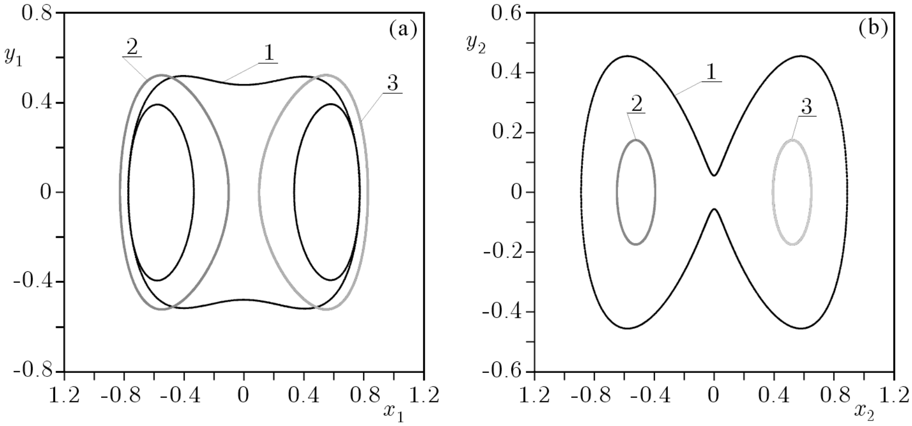

Fig. 12. Attractors of the regular response in $x_1 y_1$ (a) and $x_2 y_2$ (b) Poincaré sections; model with one self-excitation term, $\delta_{12} = 0.6$

The motion for the first and the second NNM, received after system decomposition, is represented by equations (5.3) and (5.4). Solutions to these equations in the phase plane are plotted in Fig. 13.

As results from Fig. 13a, the trajectory tends to the equilibrium position located in the left potential well, while the second mode (Fig. 13b) goes to infinity. Thus, we can conclude that the third order approximation does not represent the modal response of the system properly. The nonlinear terms, which have been truncated in (3.7) over the third order, may play an important role in the modal decomposition of the considered, very sensitive, nonlinear system with four potential wells.

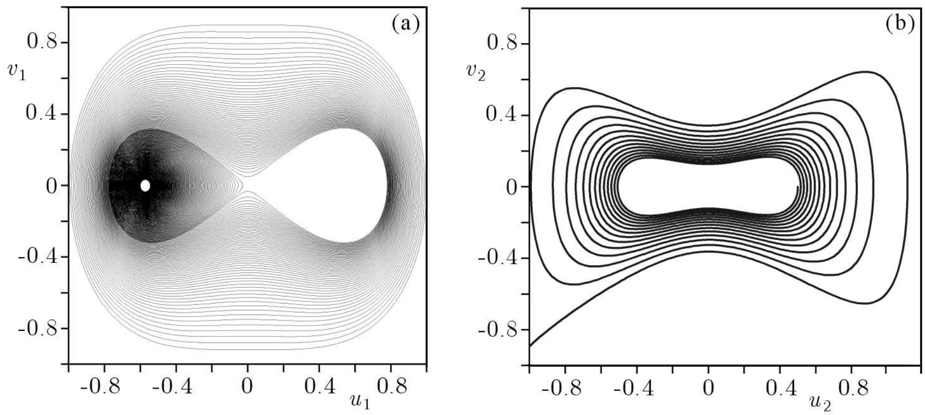

Fig. 13. Modal response of the system for $\delta_{12} = 0.6$; (a) numerical solution of Eq. (5.3) and (b) Eq. (5.4)

6. Conclusions

The analysis presented in the paper concerns dynamics of a nonlinear self-excited system. The NNMs are applied to decouple motion of the system. Using a transformation which takes into account nonlinear stiffness and nonlinear damping terms, the modal surfaces strongly dependent on the displacement and velocity, are constructed. The proposed technique allows for decoupling of the self-excitation generated by both oscillators. Two separate limit cycles have been obtained. The results in the third order approximation are in a good agreement if the system has a positively defined linear part of the stiffness matrix. However, for the system with four potential wells, which is determined by parameters with a negative linear part of the stiffness and strong nonlinear terms, the response can be regular or chaotic. Generally, in such a case, the system is very sensitive to initial conditions and parameter perturbations. The method of NNMs fails such a structure. If the response is regular, the motion can be decoupled into two modal oscillators. However, the third order approximation does not satisfy the convergence of the modal solutions. Construction of NNMs in a higher order approximation is theoretically possible but from the practical point of view rather laborious. Separation of chaotic motion on two modal oscillators can not give a proper solution because any of the separated subsystems may not produce a chaotic response due to too low (2D) dimension of the oscillator.

Acknowledgments

This work has been partially supported by the EU grant MTKD-CT-2004-014058 "*Modern Composite Materials Applied in Aerospace, Civil and Mechanical Engineering: Theoretical Modelling and Experimental Verification*".

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Nieliniowe postacie drgań sprzęgniętych oscylatorów samowzbudnych w obszarach ruchu regularnego i chaotycznego

Streszczenie

W pracy przedstawiono analizę drgań sprzęgniętych nieliniowych oscylatorów samowzbudnych. Ruch regularny układu, generowany przez nieliniowe tłumienie, określono poprzez zastosowanie nieliniowych postaci drgań. Zbadano wpływ członów nieliniowych i intensywność samowzbudzenia na odpowiedź układu oraz postacie drgań. Określono parametry układu prowadzące do ruchu chaotycznego oraz nieliniowe postacie drgań występujące w pobliżu tego obszaru. Stwierdzono, że układ autonomiczny (bez wymuszeń jawnie zależnych od czasu) o dwóch stopniach swobody, może przejść do ruchu chaotycznego jeśli posiada funkcję potencjału z tzw. "dołkami". Jednak, nieliniowe postacie drgań wyznaczone dla tego czułego regionu, w pobliżu chaosu, nie są zgodne z wynikami otrzymanymi z bezpośredniej symulacji numerycznej.

Manuscript received January 9, 2008; accepted for print January 30, 2008