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FREE AND FORCED VIBRATIONS OF TIMOSHENKO BEAMS DESCRIBED BY SINGLE DIFFERENCE EQUATION

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In the paper, a new approach to description of the Timoshenko beam free and forced vibrations by a single equation is proposed. The solution to such an equation is a function of vibration amplitudes. The boundary conditions corresponding to such a description of the beam vibration are also given.

It was proved that the form of solution to the differential equation depends on the vibration frequency. The change of the solution form occurs when the frequency crosses a specific value $\omega = \sqrt{GkA/(\rho I)}$.

The correctness of proposed description was checked through the analysis of free vibration frequencies and amplitudes of forced vibrations with different boundary conditions as well as comparison with the results of finite element analysis.

Key words: Timoshenko beam, Green function, boundary conditions, forced vibrations

1. Introduction

Beam vibrations described by the Timoshenko model have been studied over the years by many authors (Stephen and Puchegger, 1982; Kaliski, 1992; Zhu *et al.*, 2006; Si *et al.*, 2007). This has been always a description through a system of second order differential equations, in which the vibration amplitude and the angle due to pure bending were the searched functions. Boundary conditions related to the initial-boundary value problem under consideration were described by a proper differential equation of both or only one of those functions. In the book Kaliski (1992), the free vibration equation is written in the form of a single equation depending only on a single function – vibration amplitude. However, there is a remark that such a description concerns only a simply supported beam. This remark arises from the fact that in the case of the simply supported beam the boundary conditions are described by the same differential equations as in the case of the Euler-Bernoulli beam.

In this paper, a method to derive a single differential equation of the fourth order describing free and forced vibrations of a Timoshenko beam is given. In addition, the equations formed during this derivation serve to define the boundary condition equations.

It was also shown that the solution form of the vibration differential equation depends on the examined vibration frequency. The change of the solution form occurs when the frequency crosses a specific value. This value is known from literature as the cut-off frequency (Stephen and Puchegger, 2006) or critical one (Chan *et al.*, 2002).

The correctness of such a description for the Timoshenko beam vibrations was checked through the analysis of free vibration frequencies with different boundary conditions and amplitudes of vibrations excited by a harmonic force and through comparison with the results of finite element analysis.

Construction of the dynamic Green functions was proposed to solve the problem of forced vibration amplitudes of the beam, excited by an arbitrary function of time t and applied to a beam in an arbitrary way, as a function of the spatial coordinate x. This is the function of beam vibration amplitudes forced by the harmonic unit force. The methods of determining this function, described in literature (Kukla, 1997; Lueschen *et al.*, 1996), consist in defining the Green matrix. This is necessary for description of the beam vibration by a system of two equations.

The single-equation description of vibrations substantially facilitates developing the inverse models, which (for the Euler-Bernoulli beam) are used by the present author for diagnostic (Majkut, 2004, 2005a,b, 2006) and structural modification (Majkut, 2008; Majkut and Michalczyk, 2002).

2. Equations of vibration of the Timoshenko beam

The Timoshenko model is an extension of the Euler-Bernoulli model by taking into account two additional effects: shearing force effect and rotary motion effect.

In any beam except one subject to pure bending only, a deflection due to the shear stress occurs. The exact solution to the beam vibration problem requires this deflection to be considered. So, the angle $\partial y(x,t)/\partial x$ between the beam axis and x axis is a sum of the angle $\Theta(x,t)$ due to pure bending and the shear angle $\gamma(x,t)$ i.e.: $\partial y(x,t)/\partial x = \Theta(x,t) + \gamma(x,t)$.

Another factor that affects the lateral vibration of the beam, neglected in Euler-Bernoulli's model, is the fact that each section of the beam rotates slightly in addition to its lateral motion when the beam deflects. The influence of the beam section rotation is taken into account through the moment of inertia, which modifies the equation of moments acting on an infinitesimal beam element: $dM_B(x,t) = -I\rho\partial^2\Theta(x,t)/\partial t^2 dx$, where: ρ is density of the beam material, I – the second moment of area.

By applying d'Alembert's principle, the system of coupled differential equations for transverse vibration of the uniform Timoshenko beam with a constant cross section is given by

$$
\begin{aligned}
&-\frac{\partial Q(x,t)}{\partial x} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = q(x,t) \\
&-\frac{\partial M(x,t)}{\partial x} + Q(x,t) - I\rho \frac{\partial^2 \Theta(x,t)}{\partial t^2} = 0
\end{aligned}
\tag{2.1}
$$

where: $Q(x,t) = kGA\gamma(x,t)$ is the shear force, $M(x,t) = EI\partial\Theta(x,t)/\partial x$ – bending moment, k – Timoshenko shear coefficient depending on the cross-section of the beam (Stephen and Puchegger, 2006; Rubin, 2003), G – shear modulus, A – cross sectional area, EI – bending stiffness, $y(x,t)$ – vibration amplitude, $q(x,t)$ – external force.

The system of differential Eqs. (2.1) describes the Timoshenko beam vibration, where the searched functions are the vibration amplitude $y(x,t)$ and the angle due to pure bending $\Theta(x,t)$.

3. Free vibrations described by the single equation

The Fourier method of variable separation is employed to find functions satisfying system of Eqs. (2.1). It is assumed that each function $y(x,t)$ and $\Theta(x,t)$ can be presented in the form of a product of a function dependent on the spatial coordinate x and a function dependent on time t (with the same time function).

$$
y(x,t) = X(x)T(t) \qquad \Theta(x,t) = Y(x)T(t) \tag{3.1}
$$

With such an assumption, after several simple transformations of system (2.1), it can be rewritten as

$$
\begin{aligned}
&X''(x) + aX(x) - Y'(x) = 0 \\
&Y''(x) + bY(x) + cX'(x) = 0 \\
&\ddot{T}(t) + \omega^2 T(t) = 0
\end{aligned}
\tag{3.2}
$$

where

$$
a = \frac{\omega^2 \rho}{kG} \qquad\qquad b = \frac{\rho\omega^2}{E} - c \qquad\qquad c = \frac{GkA}{EI}
$$

and ω – vibration frequency.

By eliminating the function $Y(x)$ from the first two equations of system (3.2)

$$
Y(x) = -\frac{1}{b}[X'''(x) + (a+c)X'(x)] \tag{3.3}
$$

one can get an equation for the transverse displacement $X(x)$. So, the single Timoshenko beam equation of free vibration is obtained in the form

$$
X^{(4)}(x) + dX''(x) + eX(x) = 0 \tag{3.4}
$$

where

$$
d = a + b + c = \frac{\omega^2 \rho I\Big(1 + \frac{E}{Gk}\Big)}{EI} \qquad\qquad e = ab = \frac{\omega^2\Big(\omega^2\rho^2\frac{I}{Gk} - \rho A\Big)}{EI}
$$

In the next section of the paper, the solution to Eq. (3.4) will be searched.

The function $Y(x)$ (Eq. (3.3)) depends on derivatives of the vibration amplitude function $X(x)$. This equation will be used to define boundary conditions dependent only on the vibration amplitude function $X(x)$ and its derivatives.

4. Solution to the Timoshenko beam differential equation

The characteristic equation of Eq. (3.4) has the form

$$
r^4 + dr^2 + e = 0 \tag{4.1}
$$

Replacing $r^2 = \widetilde{z}$, Eq. (4.1) can be rewritten in the form

$$
\widetilde{z}^2 + d\widetilde{z} + e = 0
$$

Its roots are

$$\widetilde{z}_1 = \frac{1}{2}(-d + \sqrt{\Delta}) \qquad\qquad \widetilde{z}_2 = \frac{1}{2}(-d - \sqrt{\Delta})$$

where

$$\Delta = d^2 - 4e = \omega^4\rho^2 I^2\Big(1 - \frac{E}{kG}\Big)^2 + 4EI\omega^2\rho A$$

It is easy to observe that $\Delta > 0$ $\forall\omega$ (for all ω).

Now one should carry out a discussion about signs of the roots $\widetilde{z}_1$ and $\widetilde{z}_2$ as the frequency function

$$\begin{aligned}
&\widetilde{z}_2 < 0 \quad \forall\omega \\
&\widetilde{z}_1 > 0 \quad \Leftrightarrow \quad \sqrt{d^2 - 4e} > d \quad \Rightarrow \quad e < 0 \quad \text{for} \quad \omega^2 < \frac{GkA}{\rho I} \\
&\widetilde{z}_1 < 0 \quad \text{for} \quad \omega > \sqrt{\frac{GkA}{\rho I}}
\end{aligned}$$

Two possible solutions to Eq. (3.4) come from the above discussion:

- For $\omega < \sqrt{GkA/(\rho I)}$

  The roots of Eq. (4.1) are

  $$r_1 = \sqrt{\widetilde{z}_1} \qquad r_2 = -\sqrt{\widetilde{z}_1} \qquad r_3 = \mathrm{i}\sqrt{-\widetilde{z}_2} \qquad r_4 = -\mathrm{i}\sqrt{-\widetilde{z}_2}$$

  This gives a solution in the form

  $$X(x) = C_1 \mathrm{e}^{\sqrt{\widetilde{z}_1}x} + C_2 \mathrm{e}^{-\sqrt{\widetilde{z}_1}x} + C_3 \mathrm{e}^{\mathrm{i}\sqrt{\widetilde{z}_2}x} + C_4 \mathrm{e}^{-\mathrm{i}\sqrt{\widetilde{z}_2}x} \tag{4.2}$$

  With the use of Euler's formulae, solution (4.2) may be also expressed in the form of trigonometric and hyperbolic functions

  $$X(x) = P_1 \cosh\lambda_1 x + P_2 \sinh\lambda_1 x + P_3 \cos\lambda_2 x + P_4 \sin\lambda_2 x \tag{4.3}$$

  where

  $$\lambda_1^2 = |\widetilde{z}_1| = \frac{-d + \sqrt{\Delta}}{2} \qquad\qquad \lambda_2^2 = |\widetilde{z}_2| = \frac{d + \sqrt{\Delta}}{2}$$

- For $\omega > \sqrt{GkA/(\rho I)}$

  In this case, the roots of Eq. (4.1) are

  $$r_1 = \mathrm{i}\sqrt{-\widetilde{z}_1} \qquad r_2 = -\mathrm{i}\sqrt{-\widetilde{z}_1} \qquad r_3 = \mathrm{i}\sqrt{-\widetilde{z}_2} \qquad r_4 = -\mathrm{i}\sqrt{-\widetilde{z}_2}$$

and the solution for free vibration of a Timoshenko beam (Eq. (3.4)) for such frequencies has the form

$$X(x) = Q_1 \cos \lambda_1 x + Q_2 \sin \lambda_1 x + Q_3 \cos \lambda_2 x + Q_4 \sin \lambda_2 x \tag{4.4}$$

where

$$\lambda_1^2 = |\widetilde{z}_1| = \frac{d - \sqrt{\Delta}}{2} \qquad\qquad \lambda_2^2 = |\widetilde{z}_2| = \frac{d + \sqrt{\Delta}}{2}$$

The integration constants P_i and Q_i $(i = 1, \dots, 4)$ depend on the boundary conditions associated with the initial-boundary value problem under consideration.

5. Boundary conditions for the Timoshenko beam model

The boundary conditions are described in such a way that the letters $(\widetilde{\mathrm{a}})$ and $(\widetilde{\mathrm{b}})$ denote their physical models, and the letters (a_i) and (b_i) denote their mathematical models after separation of variables.

1. Hinged end $(x_i = 0$ or $x_i = l)$ or internal support at x_i

$$(\widetilde{\mathrm{a}}) \quad y(x_i, t) = 0 \qquad\qquad (\widetilde{\mathrm{b}}) \quad M(x_i, t) = EI\frac{\partial \Theta(x_i, t)}{\partial x} = 0$$

The conditions (a) and (b) after separation of variables and taking into account Eq. (3.3) have the forms

$$(\mathrm{a}_1) \quad X(x_i) = 0 \qquad\qquad (\mathrm{b}_1) \quad Y'(x_i) = 0 \Leftrightarrow X''(x_i) + aX(x_i) = 0$$

2. Fixed clamped end $(x_i = 0$ or $x_i = l)$

$$(\widetilde{\mathrm{a}}) \quad y(x_i, t) = 0 \qquad\qquad (\widetilde{\mathrm{b}}) \quad \Theta(x_i, t) = 0$$

after separation of variable

$$(\mathrm{a}_2) \quad X(x_i) = 0 \qquad\qquad (\mathrm{b}_2) \quad Y = 0 \Leftrightarrow X'''(x_i) + (a + c)X'(x_i) = 0$$

3. Free end $(x_i = 0$ or $x_i = l)$

$$(\widetilde{\mathrm{a}}) \quad M(x_i, t) = EI\frac{\partial \Theta(x_i, t)}{\partial x} = 0$$

$$(\widetilde{\mathrm{b}}) \quad Q(x_i, t) = GkA\Big(\frac{\partial y(x_i, t)}{\partial x} - \Theta(x_i, t)\Big) = 0$$

after separation of variables

$$(\mathrm{a}_3)\quad X''(x_i) + aX(x_i) = 0 \qquad\qquad (\mathrm{b}_3)\quad dX'(x_i) + X'''(x_i) = 0$$

4. Generally supported beam

— at $x = 0$ – Fig. 1a

$$(\widetilde{\mathrm{a}})\quad Q(0,t) = -k_T y(0,t) \qquad\qquad (\widetilde{\mathrm{b}})\quad M(0,t) = k_R \frac{\partial y(0,t)}{\partial x}$$

after separation of variable

$$\begin{aligned}&(\mathrm{a}_4)\quad GkA[(a+b+c)X'(0) - X'''(0)] + k_T X(0) = 0\\&(\mathrm{b}_4)\quad X''(0) + aX(0) - \frac{k_R}{EI}X'(0) = 0\end{aligned}$$

— at $x = l$ – Fig. 1b

$$(\widetilde{\mathrm{a}})\quad Q(l,t) = k_T y(l,t) \qquad\qquad (\widetilde{\mathrm{b}})\quad M(l,t) = -k_R \frac{\partial y(l,t)}{\partial x}$$

after separation of variable

$$\begin{aligned}&(\mathrm{a}_4)\quad GkA[(a+b+c)X'(l) - X'''(l)] - k_T X(l) = 0\\&(\mathrm{b}_4)\quad X''(l) + aX(l) + \frac{k_R}{EI}X'(l) = 0\end{aligned}$$

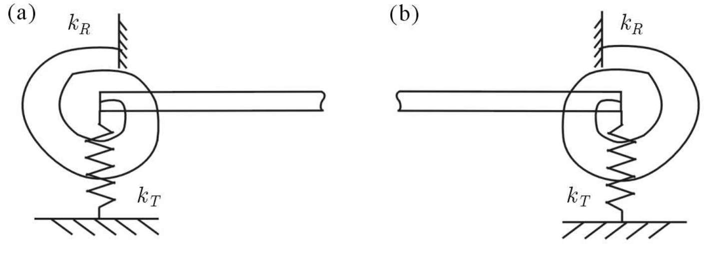

Fig. 1. General boundary conditions of the beam (a) at $x = 0$ (b) at $x = l$

The thus described boundary conditions will be used for determining natural frequencies of the beam with different boundary conditions.

6. Determination of the natural frequencies

The choice of the beam model (Euler-Bernoulli or Timoshenko), which should be used in the analysis of transverse vibrations of a given beam, depends on the ratio of its height h to half of the wavelength corresponding to the vibration frequency (the distance between two adjacent nodes l). Because the difference between the free vibration frequencies for $h/l \approx 8\%$ is equal to 5% and increases with increasing h/l, it is accepted that Timoshenko model should be employed for beams and frequencies for which this ratio is larger than 10%. Such beams are called stocky beams, while the beams for which the Euler-Bernoulli model is sufficient are called the slender ones. But the Timoshenko beam model must be taken in the analysis of high-frequency vibration of all beams.

To check correctness of the proposed description of beam vibrations, the calculations were carried out both for a slender beam (small differences in the free vibration frequencies obtained from both models of the beam) and for a stocky beam, for which these differences should be larger.

The calculations were carried out for the beam with: $E = 2.1 \cdot 10^{11}\,\mathrm{Pa}$, $G = 8.1 \cdot 10^{10}\,\mathrm{Pa}$, $\rho = 7860\,\mathrm{kg/m^3}$, length $l = 1\,\mathrm{m}$ and cross-section $b \times h = 0.02 \times 0.08\,\mathrm{m^2}$ (stocky beam) and $b \times h = 0.02 \times 0.03\,\mathrm{m^2}$ (slender beam). In order to check the correctness of free vibration frequencies calculations, FEM analysis was carried out by the author with the use of proper one-dimensional finite elements (Cheung and Leung, 1991).

6.1. Simply supported beam

The boundary conditions for a simply supported beam are: at $x = 0$, $X(0) = 0$ and $X''(0) + aX(0) = 0$; at $x = l$, $X(l) = 0$ and $X''(l) + aX(l) = 0$.

The form of the solution to the vibration equation depends on the interval to which the searched natural frequency belongs:

- For $\omega < \sqrt{GkA/(\rho I)}$, the solution to Eq. (3.4) has form (4.3).

  From the boundary conditions at $x = 0$, the following equations are obtained

  $$P_1 + P_3 = 0 \qquad\qquad \lambda_1^2 P_1 - \lambda_2^2 P_3 = 0$$

  This system of equations is satisfied when $P_1 = P_3 = 0$, or in the case when $\lambda_1^2 + \lambda_2^2 = 0$, i.e. when $\sqrt{d^2 - 4e} = 0$, and this is possible only when $\omega = 0$, which describes motion of the beam as a rigid body, what is the impossible because of the boundary conditions.

So, the boundary conditions at $x = 0$ require that $P_1 = P_3 = 0$, which corresponds to the similar solution for the Euler-Bernoulli beam.

The boundary conditions at $x = l$ are expressed by the matrix equation

$$\begin{bmatrix} \sinh\lambda_1 l & \sin\lambda_2 l \\ \lambda_1^2 \sinh\lambda_1 l & -\lambda_2^2 \sin\lambda_2 l \end{bmatrix} \begin{bmatrix} P_2 \\ P_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.1}$$

The nontrivial solution to Eq. (6.1) is obtained from the condition that the main matrix determinant is equal to zero.

- Solution to Eq. (3.4) for $\omega > \sqrt{GkA/(\rho I)}$ has form (4.4).

  From the boundary conditions at $x = 0$, the following equations are obtained:

$$Q_1 + Q_3 = 0 \qquad\qquad \lambda_1^2 Q_1 + \lambda_2^2 Q_3 = 0$$

  This system of equations is satisfied either for $Q_1 = Q_3 = 0$ or for $\lambda_1^2 - \lambda_2^2 = 0$. Fulfilling the second condition is possible only when $\omega = 0$, which describes motion of the beam as a rigid body, what is the impossible because of the boundary conditions.

  Therefore, the boundary conditions at $x = 0$ require that $Q_1 = Q_3 = 0$.

  The boundary conditions at $x = l$ are expressed in the matrix form

$$\begin{bmatrix} \sin\lambda_1 l & \sin\lambda_2 l \\ \lambda_1^2 \sin\lambda_1 l & -\lambda_2^2 \sin\lambda_2 l \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.2}$$

  The roots of the main matrix determinant are the eigenvalues of the initial-boundary value problem under consideration.

Table 1 contains the first five natural frequencies of the stocky beam together with the corresponding frequencies calculated for the Euler-Bernoulli model. In order to check the correctness of the proposed approach, the frequencies were compared with the results of FEM analysis carried out with the use of proper 1-D finite elements (Cheung and Leung, 1991).

Table 2 contains the natural frequencies for the slender beam.

The results presented in Tables 1 and 2 prove correctness of the proposed description of the Timoshenko beam vibration. The relative difference in the values of the free vibration frequencies determined with the use of different models for the slender beam is equal to 0.2% for the first free vibration frequency and increases to 5.4% for the fifth frequency. The difference increases because the wavelength decreases (the ratio h/l increases) with the growth of

Table 1. Natural frequencies of the stocky simply supported beam

	Timoshenko beam (analytical method)	E-B beam (analytical method)	Timoshenko beam (FEM)	E-B beam (FEM)
1	1159.4	1178.1	1160.5	1178.1
2	4436.8	4712.6	4447.4	4712.6
3	9357.6	10603.3	9386.1	10603.0
4	15410.0	18850.0	15439.0	18850.0
5	22163.0	29453.5	22183.0	29454.0

Table 2. Natural frequency of the slender simply supported beam

	Timoshenko beam (analytical method)	E-B beam (analytical method)	Timoshenko beam (FEM)	E-B beam (FEM)
1	440.8	441.8	440.8	441.8
2	1751.3	1767.2	1752.2	1767.2
3	3897.4	3976.2	3901.2	3978.2
4	6825.5	7068.8	6736.9	7068.9
5	10473.2	11045.0	10494.0	11045.0

the free vibration frequency. In the case of the stocky beam, the differences in the determined frequencies are from 1.6% up to 33%. In both cases. there are no significant differences in the frequency values obtained from the proposed description and from the FEM analysis.

6.2. Cantilever beam

Boundary conditions for the cantilever beam are: at $x = 0$, $X(0) = 0$ and $X'''(0)+(a+c)X'(0) = 0$; at $x = l$, $dX'(l)+X'''(l) = 0$ and $X''(l)+aX(l) = 0$.

The form of the solution for the free vibration problem depends on the interval to which the searched natural frequency belongs:

- For $\omega < \sqrt{GkA/(\rho I)}$, the solution to Eq. (3.4) has form (4.3), the boundary conditions are expressed by a matrix equation

$$\mathbf{AC} = \mathbf{0}$$

where

$$
\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1(\lambda_1^2 + a + c) & 0 & a_{24} \\ (\lambda_1^2 - d)\lambda_1 \sinh \lambda_1 l & (\lambda_1^2 - d)\lambda_1 \cosh \lambda_1 l & (\lambda_2^2 + d)\lambda_2 \sin \lambda_2 l & a_{34} \\ (\lambda_1^2 + a) \cosh \lambda_1 l & (\lambda_1^2 + a) \sinh \lambda_1 l & (-\lambda_2^2 + a) \cos \lambda_2 l & a_{44} \end{bmatrix}
$$

$$
\mathbf{C}^\top = [P_1, P_2, P_3, P_4]
$$

and

$$
a_{24} = \lambda_2(-\lambda_2^2 + a + c) \qquad a_{34} = (-\lambda_2^2 - d)\lambda_2 \cos \lambda_2 l
$$
$$
a_{44} = (-\lambda_2^2 + a) \sin \lambda_1 l
$$

The coefficients a, b, c, d and e are defined in Section 3.

Natural frequencies of the beam are determined from the equation: $\det \mathbf{A} = 0$.

- For frequencies $\omega > \sqrt{GkA/(\rho I)}$ (the so-called second spectrum of the beam (Stephen and Puchegger, 1982)), the solution has form (4.4), and the main matrix is

$$
\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1(-\lambda_1^2 + a + c) & 0 & a_{24} \\ (\lambda_1^2 + d)\lambda_1 \sin \lambda_1 l & (-\lambda_1^2 - d)\lambda_1 \cos \lambda_1 l & (\lambda_2^2 + d)\lambda_2 \sin \lambda_2 l & a_{34} \\ (-\lambda_1^2 + a) \cos \lambda_1 l & (-\lambda_1^2 + a) \sin \lambda_1 l & (-\lambda_2^2 + a) \cos \lambda_2 l & a_{44} \end{bmatrix}
$$

Table 3 contains the first five natural frequencies of the cantiilever beam together with the corresponding frequencies calculated for the Euler-Bernoulli model of the beam.

Table 3. Natural frequency of the stocky cantilever beam

	Timoshenko beam (analytical method)	E-B beam (analytical method)	Timoshenko beam (FEM)	E-B beam (FEM)
1	424.1	419.6	424.1	419.3
2	2653.7	2666.8	2656.8	2630.6
3	7145.3	7387.3	7148.1	7367.6
4	13016.0	14284.0	13024.0	14443.0
5	19645.0	23383.0	19653.0	23886.0

Table 4 contains results obtained for the slender beam.

The results presented in Tables 3 and 4 prove correctness of the proposed description of the Timoshenko beam vibration and the boundary condition equations.

Table 4. Natural frequency of the slender cantilever beam

	Timoshenko beam (analytical method)	B-E beam (analytical method)	Timoshenko beam (FEM)	E-B beam (FEM)
1	157.6	153.6	159.2	154.7
2	987.7	962.6	993.5	986.5
3	2752.5	2695.3	2761.4	2762.8
4	5344.2	5281.6	5350.0	5416.0
5	8716.1	8730.9	8741.1	8957.4

7. Forced vibrations

The forced vibrations for the Timoshenko beam model, written in the form of two differential equations, are as follows

$$\begin{aligned}&\rho A\frac{\partial^2 y(x,t)}{\partial t^2}-GkA\Big(\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{\partial\Theta(x,t)}{\partial x}\Big)=q(x,t)\\&EI\frac{\partial^2\Theta(x,t)}{\partial x^2}+GkA\Big(\frac{\partial y(x,t)}{\partial x}-\Theta(x,t)\Big)-I\rho\frac{\partial^2\Theta(x,t)}{\partial t^2}=0\end{aligned}$$

After several transformations, similar to transformation of the homogeneous equation, it is possible to obtain an equation for the Timoshenko beam forced vibrations in a form dependent only on the function of the displacement $y(x,t)$

$$\begin{aligned}&EI\frac{\partial^4 y(x,t)}{\partial x^4}-\Big(\frac{EI\rho}{Gk}+I\rho\Big)\frac{\partial^4 y(x,t)}{\partial x^2\partial t^2}+\frac{I\rho^2}{Gk}\frac{\partial^4 y(x,t)}{\partial t^4}+\rho A\frac{\partial^2 y(x,t)}{\partial t^2}=\\&=q(x,t)-\frac{EI}{GkA}\frac{\partial^2 q(x,t)}{\partial x^2}+\frac{I\rho}{GkA}\frac{\partial^2 q(x,t)}{\partial t^2}\end{aligned}\tag{7.1}$$

This is the equation of transverse vibrations of the Timoshenko beam with an arbitrary exciting function $q(x,t)$ (the excitation is an arbitrary function of time t and spatial coordinate x). It is impossible to find its solution in the general case.

In the paper, a different approach to the problem of solution for the beam forced vibrations is proposed. In the proposed method, the beam can be excited by an arbitrary function of time t applied to the beam in an arbitrary way, as a function of the spatial coordinate x.

The problem of time variability of the exciting function is proposed to be solved in the following way: to transform the excitation function from the time domain into frequency domain, then to find the system response in the frequency domain and employ the inverse Fourier transform, finding in this way the vibration amplitudes function $y(x,t)$ in the time domain.

The spatial distribution of the forcing function can be taken into account by using an integral equation, i.e. by using the superposition rule. The kernel of the integral equation is the Green function.

It comes, form the above discussion, that finding the dynamic Green function is enough to find the forced vibrations function $y(x,t)$ of the beam with an arbitrary excitation function.

The dynamic Green function is a function of the beam vibration amplitudes excited by the unit harmonic force.

7.1. Dynamic Green function

To determine the dynamic Green function, one should find the solution to inhomogeneous differential equation (7.1), in which the excitation function has the form $q(x,t) = \mathit{1} \exp(\mathrm{i}\omega_w t)\delta(x-x_f)$, where $\delta(x-x_f)$ is the Dirac delta function and $\mathit{1}$ – unit excitation. In such a case, the function being its solution in the steady state, can be expressed in the form $y(x,t) = G(x,x_f)\exp(\mathrm{i}\omega_w t)$.

The function $G(x,x_f)$ is the searched dynamic Green function, in which x_f is the coordinate of the point where the force is applied to the beam.

Comment: In the paper, both internal and external damping of the beam was neglected. This is not a problem in the case of determining the eigenfrequencies and eigenvectors. In the case of forced vibration, especially of high frequency, the negligence of damping leads to large errors in the determined vibration amplitudes. Because of that, the solution to the forced vibration equation was searched only for frequencies below the cut-off frequency, i.e. for $\omega < \sqrt{GkA/(\rho I)}$.

The function $G(x,x_f)$ will be determined as a sum of the general solution to the related homogeneous equation and the particular solution

$$G(x,x_f) = G_0(x) + G_1(x,x_f)H(x-x_f) \tag{7.2}$$

The function $G_0(x)$ is the solution to the homogeneous equation in form (4.3), for $x \in (0,l)$, where $G_1(x,x_f)$ is the solution to the inhomogeneous equation determined for $x > x_f$. $H(x-x_f)$ is the step function.

The solution to the inhomogeneous equation $G_1(x,x_f)$ can be expressed in the form

$$\begin{aligned} G_1(x,x_f) = R_1\cosh[\lambda_1(x-x_f)] + R_2\sinh[\lambda_1(x-x_f)] + \\ +R_3\cos[\lambda_2(x-x_f)] + R_4\sin[\lambda_2(x-x_f)] \end{aligned}$$

The constants R_i were determined from the continuity conditions and the jump condition at $x = x_f$:

$$
\begin{aligned}
&G(x_f^+, x_f) - G(x_f^-, x_f) = 0 \qquad && \Theta(x_f^+) - \Theta(x_f^-) = 0\\
&M(x_f^+) - M(x_f^-) = 0 && Q(x_f^+) - Q(x_f^-) = 1
\end{aligned}
$$

The relationships between the individual physical quantities and the Green function are given in the description of the boundary conditions (see Section 5).

The constants of integration, determined from the continuity conditions, are: $R_1 = R_3 = 0$

$$
\begin{aligned}
R_2 &= 1\frac{b}{GkA}\frac{\lambda_2^2 - a - c}{\lambda_1(\lambda_1^2 + \lambda_1^2)(a + c + d)}\\
R_4 &= 1\frac{b}{GkA}\frac{\lambda_1^2 + a + c}{\lambda_2(\lambda_1^2 + \lambda_1^2)(a + c + d)}
\end{aligned}
$$

Finally, the dynamic Green function for the Timoshenko beam has the form

$$
\begin{aligned}
G(x, x_f) = P_1 \cosh\lambda_1 x + P_2 \sinh\lambda_1 x + P_3 \cos\lambda_2 x + P_4 \sin\lambda_2 x +\\
+R_2 \sinh[\lambda_1(x - x_f)]H(x - x_f) + R_4 \sin[\lambda_2(x - x_f)]H(x - x_f)
\end{aligned}
\tag{7.3}
$$

In Fig. 2, one can see the functions of amplitudes for the forced vibrations determined for the simply supported beam with: $E = 2.1 \cdot 10^{11}$ Pa, $G = 8.1 \cdot 10^{10}$ Pa, $\rho = 7860\mathrm{kg/m}^3$, length $l = 1$m and cross-section $b \times h = 0.02 \times 0.08$ (stocky beam) – Fig. 2a and $b \times h = 0.02 \times 0.03$ (slender beam) – Fig. 2b. In both figures, the solid line (—) denotes the amplitude for the Timoshenko beam model, while the dashed line (- -) denotes the amplitude for the Euler-Bernoulli model of beam. There are no significant differences in values of vibration amplitudes obtained from the proposed description and FEM analysis.

The functions of amplitudes for the forced vibrations determined for the cantilever beam with the same data are shown in Fig. 3, those determined for the stocky beam are shown in Fig. 3a and those for the slender beam – in Fig. 3b. In both figures, the solid line (—) denotes the amplitude for the Timoshenko beam model, while the dashed line (- -) denotes the amplitude for the Euler-Bernoulli model of beam. There are no significant differences in values of vibration amplitudes obtained from the proposed description and FEM analysis.

The graphs of the harmonically excited vibration amplitudes, shown in Fig. 2 and Fig. 3 prove correctness of the proposed description of the Timoshenko beam vibration and the boundary condition equations.

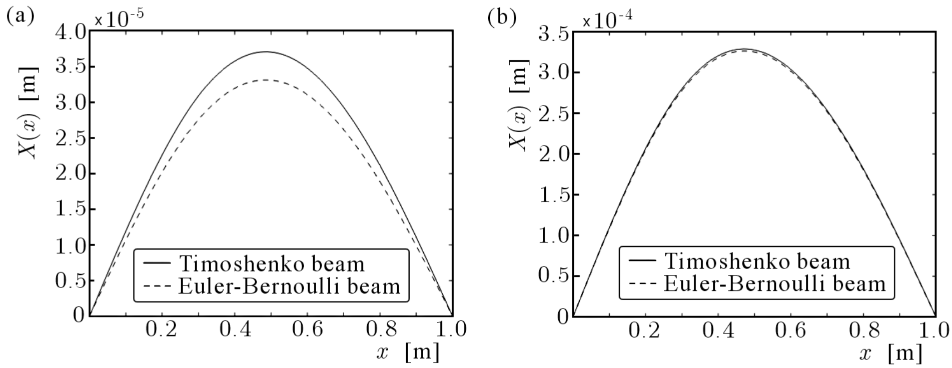

Fig. 2. Vibration amplitudes of the simply supported beam (a) stocky beam (b) slender beam

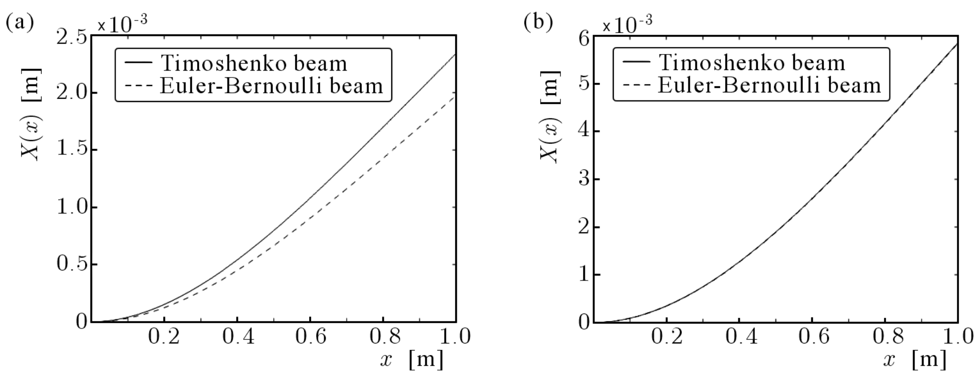

Fig. 3. Vibration amplitudes of the cantilever beam (a) stocky beam (b) slender beam

8. Summary

In the paper, the method to derive a single equation for free and forced vibrations of the Timoshenko beam model was proposed. The solution to such an equation is a function of vibration amplitudes. Searching for the function describing the angle due to pure bending, necessary in descriptions of the Timoshenko beam vibration known so far in the literature, is unnecessary when using such an approach. The boundary conditions corresponding to such a description of the beam vibration were also given.

It was proved that the form of solution to the differential equation depends on the examined vibration frequencies. The change of the solution form occurs when the frequency crosses a specific value, determined in the paper, which depends on the material properties and the beam cross-section $\omega = \sqrt{GkA/(\rho I)}$.

The correctness of such a description of the Timoshenko beam vibrations was checked by analysis of free vibrations of the beam with different boundary conditions. The results presented in Tables 1-4 indicate correctness of the proposed method. The maximal differences between the values of free vibration frequencies determined with the use of the Timoshenko beam model and those determined with the use of the Euler-Bernoulli beam model are 33% for the stocky beam and 5.4% for the slender one. There are no significant differences in the frequency values obtained from the proposed description and FEM analysis.

Construction of the dynamic Green functions was proposed to solve the problem of vibration amplitudes excited by an arbitrary function of time t and applied to the beam in an arbitrary way, as a function of the spatial coordinate x. This is a function of beam vibration amplitudes forced by the harmonic unit force.

Knowing the dynamic Green function, one can determine the frequency response function of the system (i.e. system transmittance) and next, by employing the inverse Fourier transform, the impulse response function.

The dynamic response of the beam to the point force with an arbitrary change in time can be found by using convolution of this time function and the impulse response of the beam.

Another possibility is to transform the time function of the excitation from the time domain into the frequency domain and next to find the system response in the frequency domain for the excitation by each of the frequency component. For this purpose, the dynamic Green function can be used (the frequencies and the corresponding amplitudes of the excitation are known). Next, the inverse Fourier transform can be employed to obtain in this way the vibrations in the time domain.

The spatial distribution of the excitation can be taken into account by using the superposition rule, i.e. the beam integral equation. The kernel of the integral equation is the Green function.

Form the above discussion, it can be conducted that finding the dynamic Green function is enough to find the forced vibrations function of time t and spatial coordinate x i.e. the function $y(x,t)$ due to an arbitrary excitation function.

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Drgania własne i wymuszone belki Timoshenki opisane jednym równaniem różniczkowym

Streszczenie

W pracy zaproponowano, nowe podejście do opisu drgań własnych i wymuszonych belki Timoshenki przez jedno równanie różniczkowe. Rozwiązaniem takiego równania jest funkcja amplitud drgań. Podano również równania opisujące warunki brzegowe odpowiednie do takiego opisu drgań.

Udowodniono, że forma rozwiązania równania różniczkowego zależy od analizowanej częstości drgań. Zmiana formy rozwiązania zmienia się, gdy częstość osiąga określoną w pracy wartość $\omega = \sqrt{GkA/(\rho I)}$.

Poprawność zaproponowanego opisu sprawdzono przez analizę częstości drgań własnych i amplitudy drgań wymuszonych belek z różnymi warunkami brzegowymi i porównaniem z wynikami otrzymanymi z analizy MES belki.

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