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SENSITIVITY ANALYSIS OF AN IDENTIFICATION METHOD DEDICATED TO NONLINEAR SYSTEMS WORKING UNDER OPERATIONAL LOADS

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In the paper, the exploitational nonlinear systems identification method based on algorithms of the restoring force, boundary perturbations and direct parameter identification methods is presented. The obtained parameter estimates provide information concerning forces transferred on the foundation and find application in the model-based diagnostics.
The results of the sensitivity analysis carried out in order to assess the influence of input parameters uncertainties (accuracy of resonant frequency and amplitude estimates, errors of transfer function estimation in operational conditions, value of introduced additional mass) on the accuracy of estimated system parameters are also presented.

Key words: sensitivity analysis, nonlinear system identification, operational loads

1. Introduction

Each real mechanical structure is nonlinear to some degree. Typical sources of nonlinearities are of geometrical origin (Kerschen *et al.*, 2006; Nayfeh and Pai, 2004) resulting from considerable structure deformations, physical nonlinearities (Kerschen *et al.*, 2006; Schultze *et al.*, 2001) related to nonlinear material properties, nonlinear damping forces (Al-Bender *et al.*, 2004) deriving from energy dissipation phenomena (e. g. dry, internal friction), nonlinear boundary conditions (Babitsky and Krupenin, 2001; Kerschen *et al.*, 2006) related to appearance of clearances and structural nonlinearities arising from application of structural elements of discrete nonlinear characteristics, such as springs and absorbers.

Although the sources of nonlinear system properties can vary, all the nonlinear systems have some common properties. In general, they do not follow the superposition principle and exhibit complex phenomena unusual for linear systems, such as jumps, self-excited and chaotic vibrations, changes in natural frequencies resulting from changes in the excitation amplitudes, co-existence of many stable equilibrium positions. In view of these properties, classical identification methods can not be used for the purposes of nonlinear system identification. It is also impossible to formulate a general identification method applicable to all nonlinear systems in all instances.

For many years, linearization methods were the only methods used for the purposes of nonlinear system identification. The most frequently used methods were the equivalent (Nichols *et al.*, 2004) and stochastic linearization. In the following years, the concept of nonlinear normal modes was introduced (Rand, 1974; Rosenberg, 1962), for weakly nonlinear systems the perturbation theory was developed (Kevorkian and Cole, 1996; Nayfeh, 1981; O'Maley, 1991). Later publications (Chan *et al.*, 1996; Chen and Cheung, 1996; Qaisi and Kilani, 2000) were dedicated to identification of strongly nonlinear systems. Recently, the researchers have been taking interest in making use of nonlinear system properties instead of avoiding or ignoring them (Nichols *et al.*, 2004; Rhoads *et al.*, 2005). More frequently, the machines and mechanical systems are designed for work in nonlinear ranges of dynamic characteristics taking advantage of phenomena characteristic for nonlinear systems.

2. Nonlinear system identification methods

The first research into nonlinear system identification methods goes back to the seventies of the last century (Ibanez, 1973; Masri and Caughey, 1979). Later works consider identification of single degree-of-freedom systems with various types of nonlinearities. Multiple degree-of-freedom identification methods are relatively new, since they have been elaborated over the last 15 years.

Independently of the applied identification method, the nonlinear system identification can be considered as a complex process consisting of nonlinearity detection, determination of the nonlinearity location, type and functional form, model parameters estimation as well as verification and validation of the estimated model.

2.1. Classical methods

Classical nonlinear system identification procedures consist of two main stages. In the first step, linear system parameters are estimated by exciting the system at an operating point where the system dynamic behaviour is nominally linear. In the second step, on the basis of nominally linear parameters found in the first step, estimation of nonlinear system parameters is performed. Classical nonlinear system identification methods can be classified according to the following categories: linearization methods, time domain methods, frequency domain methods, modal methods, time-frequency analysis methods, methods based on neural networks, wavelet transform methods, structural model updating. Such a classification is certainly not exhaustive and the additional categories can be introduced. For instance, it is possible to make distinction between parametric and nonparametric methods, single and multiple input methods, single and multi degree-of-freedom methods, etc.

Classical nonlinear system identification procedures require an input measurement or at least estimate, which can be treated as an essential disadvantage. In many mechanical systems, measurement of exciting forces (e.g. tire-road or wheel-rail contact forces) is difficult or impossible to carry out. Moreover, behaviour of a large variety of mechanical systems is not linear in a broad enough frequency range around any operating point.

2.2. Operational nonlinear system identification method

Contrary to the classical nonlinear system identification methods, the algorithm of the operational nonlinear system identification method (Fig. 1), considered in this paper, requires neither input measurement (estimate) nor linear system behaviour in a broad frequency range around an operating point (Haroon *et al.*, 2005; Iwaniec, 2009b). Therefore, it is a method convenient for parameter identification of strongly nonlinear systems working under operational loads the measurement of which is difficult or impossible to carry out. The method can be used for both nonlinearity detection and system parameter identification.

The algorithm of the method, presented schematically in Fig.1, consists in sequential application of the restoring force, boundary perturbation and direct parameter identification techniques (Haroon *et al.*, 2005).

In the first step of the algorithm, the discrete model of the considered real system is assumed and dynamic equations of motion for individual degrees-of-freedom are formulated

$$\mathbf{M}\ddot{\boldsymbol{x}}(t) + \mathbf{C}\dot{\boldsymbol{x}}(t) + \mathbf{K}\boldsymbol{x}(t) = \boldsymbol{F}_z(\boldsymbol{x}(t), t) + \boldsymbol{N}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) \tag{2.1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are mass, damping, stiffness matrices, $\boldsymbol{x}(t)$, $\dot{\boldsymbol{x}}(t)$, $\ddot{\boldsymbol{x}}(t)$ – time histories of displacements, velocities and accelerations of system masses, respectively, $\boldsymbol{F}_z$ – reduced force, $\boldsymbol{N}$ – nonlinear restoring force.

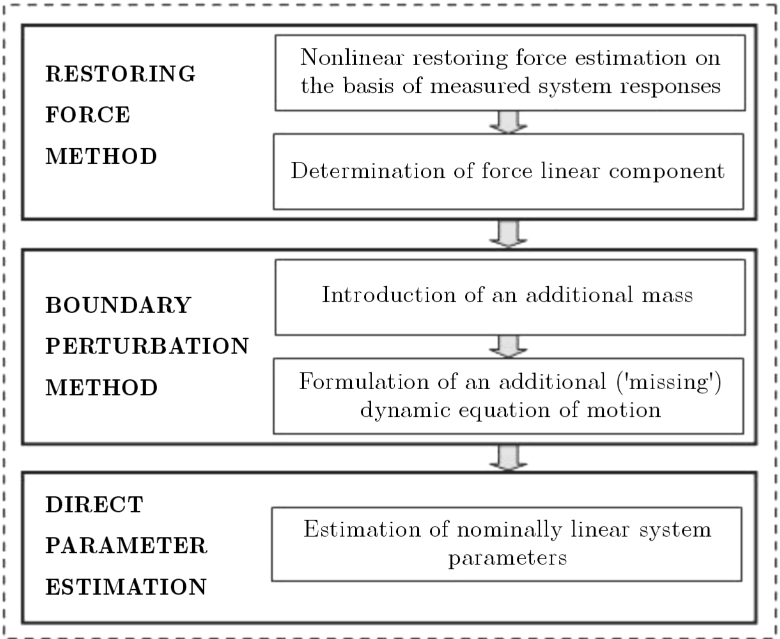

Fig. 1. Proposed and verified (Iwaniec, 2009b) identification method of nonlinear systems working under operational loads

For the purposes of research into properties of vehicle suspension systems or mechanical systems of two "dominant" degrees of freedom, the model presented in Fig. 2 can be used. That universal model, making it possible to implement the idea of the *sky-hooked-damper* (Haaron *et al.*, 2005), was applied by the author for parameter identification of the Skytruck airplane suspension system (Iwaniec, 2009b; Iwaniec and Uhl, 2007), vibratory machine body suspension system (Iwaniec, 2009a,b) and rotational machine shaft support (Iwaniec, 2007, 2009b). Dynamic equations of motion formulated for the particular model masses are as follows

$$\begin{aligned} &M_1\ddot{\boldsymbol{x}}_1 + (C_1 + C_2)\dot{\boldsymbol{x}}_1 - C_2\dot{\boldsymbol{x}}_2 + (K_1 + K_2)\boldsymbol{x}_1 - K_2\boldsymbol{x}_2 + \boldsymbol{N}_1 + \boldsymbol{N}_2 = \\ &\qquad = C_1\dot{\boldsymbol{x}}_b + K_1\boldsymbol{x}_b \\ &M_2\ddot{\boldsymbol{x}}_2 - C_2\dot{\boldsymbol{x}}_1 + C_2\dot{\boldsymbol{x}}_2 - K_2\boldsymbol{x}_1 + (K_2 + K_3)\boldsymbol{x}_2 = \boldsymbol{N}_1 \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \boldsymbol{N}_1 &= \boldsymbol{N}_1(\boldsymbol{x}_1(t), \boldsymbol{x}_2(t), \dot{\boldsymbol{x}}_1(t), \dot{\boldsymbol{x}}_2(t)) \\ \boldsymbol{N}_2 &= \boldsymbol{N}_2(\boldsymbol{x}_1(t), \boldsymbol{x}_b(t), \dot{\boldsymbol{x}}_1(t), \dot{\boldsymbol{x}}_b(t)) \end{aligned}$$

and M_1 – unsprung mass, M_2 – sprung mass, K_1 – tire stiffness coefficient, K_2 – suspension stiffness coefficient, C_1 – tire damping coefficient, C_2 – suspension damping coefficient, $\boldsymbol{x}_1$ – displacement of mass M_1, $\boldsymbol{x}_2$ – displacement of mass M_2, $\boldsymbol{x}_b$ – tire patch displacement, $\boldsymbol{N}_1$ – nonlinear restoring force acting on the vehicle suspension, $\boldsymbol{N}_2$ – nonlinear restoring force acting on the vehicle tire.

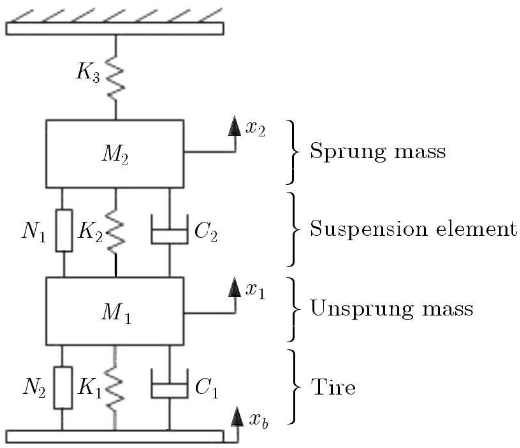

Fig. 2. Discrete model of a 2 degree-of-freedom nonlinear system (vehicle suspension system)

After transformation of the equation of motion formulated for a given degree-of-freedom into the form (2.3) and substitution of the measured system responses (usually system vibration accelerations), reconstruction of the restoring forces acting on the system of interest is performed. In case of 2 degree-of-freedom systems, equation of motion for mass M_2 (e.g. car body, railway car body) is transformed to the following form

$$M_2\ddot{\boldsymbol{x}}_2 = -C_2(\dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_1) - K_2(\boldsymbol{x}_2 - \boldsymbol{x}_1) - K_3\boldsymbol{x}_2 + \boldsymbol{N}_1 \tag{2.3}$$

where

$$\begin{array}{ll} \boldsymbol{N}_1 = \boldsymbol{N}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2) & \qquad \ddot{\boldsymbol{x}}_2 = \ddot{\boldsymbol{x}}_2(t) \\ \dot{\boldsymbol{x}}_2 = \dot{\boldsymbol{x}}_2(t) & \qquad \dot{\boldsymbol{x}}_1 = \dot{\boldsymbol{x}}_1(t) \\ \boldsymbol{x}_2 = \boldsymbol{x}_2(t) & \qquad \boldsymbol{x}_1 = \boldsymbol{x}_1(t) \end{array}$$

that makes it possible to determine relation between the acceleration of the sprung mass and difference between velocities of the system masses (relative velocity) as well as relation between acceleration of the sprung mass and difference between the displacements of system masses (relative displacement).

If the determined restoring forces are nonlinear, in the following step of the algorithm, they are approximated with the use of parametrical model and subtracted from the overall force of resistance in the system. Such a procedure makes it possible to determine the linear component of reactions

$$\ddot{\boldsymbol{x}}_2 - \boldsymbol{f}_{n1} - \boldsymbol{f}_{n2} = \frac{C_2}{M_2}(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2) + \frac{K_2}{M_2}(\boldsymbol{x}_1 - \boldsymbol{x}_2) - \frac{K_3}{M_2}\boldsymbol{x}_2 \tag{2.4}$$

where: $\boldsymbol{f}_{n1}$, $\boldsymbol{f}_{n2}$ are functions approximating the identified nonlinear restoring forces.

On this basis, with the use of the direct parameter estimation method, the parameters of the considered system are estimated. Since the measurements of system responses are carried out in operational conditions, the force exciting the system remains unknown. Therefore, the number of unknowns is higher than the number of dynamic equations of motion that can be formulated and the direct parameter estimation method provides only the ratios of system parameters.

In order to determine absolute values of the system parameters, the boundary perturbation method consisting in modification of the system dynamic properties, is applied. In practice, introduction of an additional mass is the most convenient method of structural modification. Application of such an approach makes it possible to formulate an additional equation of motion and, what follows, to determine values of the demanded parameters.

In case of a 2 degree-of-freedom system, for which the discrete model presented in Fig. 2 was used, application of the direct parameter identification method results in formulation of the following equations

$$\begin{aligned} &\boldsymbol{T}_{21}(\mathrm{i}\omega) = \frac{\boldsymbol{X}_2(\mathrm{i}\omega)}{\boldsymbol{X}_1(\mathrm{i}\omega)} \Rightarrow K_2\Big(1 - \frac{1}{T_{21}(\omega_k)}\Big) + K_3 = \omega_k^2 M_2 \qquad k = 1,2,\dots,N_f \\ &T_{21}(0) = \frac{K_2}{K_2 + K_3} \end{aligned} \tag{2.5}$$

where $\boldsymbol{T}_{21}(\mathrm{i}\omega)$ is the transfer function between displacements of masses M_2 and M_1 (determined after elimination of nonlinear restoring forces), $\boldsymbol{X}_1(\mathrm{i}\omega)$, $\boldsymbol{X}_2(\mathrm{i}\omega)$ – Fourier transform of signals $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$, respectively, N_f – number of system degrees of freedom, $T_{21}(0)$ – transfer function valuated for $\omega = 0$.

In practice, the exact values of the considered system masses remain unknown. Therefore, the direct parameter estimation method makes it possible to formulate 2 equations with 3 unknowns – M_2, K_2, K_3. An additional dynamic equation of motion, formulated for the modified system (according to the algorithm of the boundary perturbation method) by introduction of an additional mass $\varDelta M_2$, has the following form

$$\boldsymbol{T}'_{21}(\mathrm{i}\omega) = \frac{\boldsymbol{X}'_2(\mathrm{i}\omega)}{\boldsymbol{X}'_1(\mathrm{i}\omega)} \Rightarrow K_2\Big(1 - \frac{1}{T'_{21}(\omega_p)}\Big) + K_3 = \omega_p^2 M_2 \qquad p = 1,2,\dots,N'_f \tag{2.6}$$

where $T'_{21}(\mathrm{i}\omega)$ is the transfer function between displacements of masses $(M_2 + \varDelta M_2)$ and M_1, N'_f – number of the system degrees of freedom.

The absolute values of demanded parameters M_2, K_2, K_3 can be determined by solving a set of equations (2.5) and (2.6).

2.3. Sensitivity analysis

In order to assess the accuracy of the discussed operational nonlinear system identification method and the influence of the input variables measurement errors on the accuracy of estimated parameters, it was necessary to elaborate an analytical model of the system and determine the expected values of quantities $\widehat{x}$ and $\widehat{y}$ analytically on the basis of theoretical (infinitely accurate) data. In case of the considered 2 degree-of-freedom system (Fig. 2) excited to vibrations in one direction by base motion (kinematical excitation), dynamic equations of motion are as follows

$$\begin{aligned}
&M_1\ddot{\boldsymbol{x}}_1 + C_1(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2) + K_1(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{N}_1(\boldsymbol{x}_1 - \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2) = \boldsymbol{0}\\
&M_2\ddot{\boldsymbol{x}}_2 + C_1(\dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_1) + K_1(\boldsymbol{x}_2 - \boldsymbol{x}_1) + C_2\dot{\boldsymbol{x}}_2 + K_2\boldsymbol{x}_2 =\\
&\qquad = \boldsymbol{N}_2(\boldsymbol{x}_1 - \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2) + K_2\boldsymbol{x}_2 + C\dot{\boldsymbol{x}}_2 = \boldsymbol{N}_2
\end{aligned} \tag{2.7}$$

Taking into account (Goliński, 1979)

$$\begin{array}{ll}
\sqrt{\dfrac{K_1}{M_1}} = \varOmega_1 & \qquad \sqrt{\dfrac{K_2}{M_2}} = \varOmega_2\\
C_1 = 2\gamma_1 M_1 \varOmega_1 & \qquad C_2 = 2\gamma_2 M_2 \varOmega_2
\end{array} \tag{2.8}$$

and

$$\begin{array}{ll}
\dfrac{\varOmega_1}{\varOmega_2} = \varLambda & \qquad \dfrac{M_1}{M_2} = V\\
\dfrac{\omega}{\varOmega_2} = \mu_2 & \qquad \dfrac{\omega}{\varOmega_1} = \mu_1 = \dfrac{\mu_2}{\varLambda}
\end{array} \tag{2.9}$$

where $\varOmega_1$, $\varOmega_2$ are undamped natural frequencies of masses M_1 and M_2, γ – dimensionless coefficient of linear (viscotic) damping, μ – frequency ratio $(\mu = \omega/\varOmega)$, relations (2.7) can be written in the following form

$$\begin{aligned}
&\ddot{\boldsymbol{x}}_1 + 2\gamma_1\varOmega_1(\dot{\boldsymbol{x}}_1 - \dot{\boldsymbol{x}}_2) + \varOmega_1^2(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{0}\\
&\ddot{\boldsymbol{x}}_2 + 2\gamma_1 V\varOmega_1(\dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_1) + \varOmega_1^2 V(\boldsymbol{x}_2 - \boldsymbol{x}_1) + 2\gamma_2\varOmega_2\dot{\boldsymbol{x}}_2 + \varOmega_2^2\boldsymbol{x}_2 = \boldsymbol{N}_2
\end{aligned} \tag{2.10}$$

Using the Euler substitution

$$\boldsymbol{x}_1 = A\mathrm{e}^{\mathrm{i}\omega} \qquad\qquad \boldsymbol{x}_2 = B\mathrm{e}^{\mathrm{i}\omega} \tag{2.11}$$

velocities and accelerations of the considered system masses can be written in the following form

$$\begin{aligned} \dot{x}_1 &= \mathrm{i}\omega A\mathrm{e}^{\mathrm{i}\omega} & \dot{x}_2 &= \mathrm{i}\omega B\mathrm{e}^{\mathrm{i}\omega} \\ \ddot{x}_1 &= -\omega^2 A\mathrm{e}^{\mathrm{i}\omega} & \ddot{x}_2 &= -\omega^2 B\mathrm{e}^{\mathrm{i}\omega} \end{aligned} \tag{2.12}$$

Then set of equations (2.10) can be given as

$$\begin{aligned} &-\omega^2 A\mathrm{e}^{\mathrm{i}\omega} + 2\mathrm{i}\omega\gamma_1\Omega_1(A-B)\mathrm{e}^{\mathrm{i}\omega} + \Omega_1^2(A-B)\mathrm{e}^{\mathrm{i}\omega} = 0 \\ &-\omega^2 B\mathrm{e}^{\mathrm{i}\omega} + 2\mathrm{i}\omega\gamma_1 V\Omega_1(B-A)\mathrm{e}^{\mathrm{i}\omega} + \Omega_1^2 V(B-A)\mathrm{e}^{\mathrm{i}\omega} + 2\mathrm{i}\omega\gamma_2\Omega_2 B\mathrm{e}^{\mathrm{i}\omega} + \\ &\qquad + \Omega_2^2 B\mathrm{e}^{\mathrm{i}\omega} = N_2 \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} &-\omega^2 A + 2\mathrm{i}\omega\gamma_1\Omega_1(A-B) + \Omega_1^2(A-B) = N_1 \\ &-\omega^2 B + 2\mathrm{i}\omega\gamma_1 V\Omega_1(B-A) + \Omega_1^2 V(B-A) + 2\mathrm{i}\omega\gamma_2\Omega_2 B + \Omega_2^2 B = N_2 \end{aligned} \tag{2.14}$$

Below, set of equations (2.14) is presented in the matrix form

$$\begin{bmatrix} -\omega^2 + 2\mathrm{i}\omega\gamma_1\Omega_1 + \Omega_1^2) & -2\mathrm{i}\omega\gamma_1\Omega_1 - \Omega_1^2 \\ -2\mathrm{i}\omega\gamma_1 V\Omega_1 - \Omega_1^2 V & -\omega^2 + 2\mathrm{i}\omega\gamma_1 V\Omega_1 + \Omega_1^2 V + 2\mathrm{i}\omega\gamma_2\Omega_2 + \Omega_2^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \\ = \begin{bmatrix} 0 \\ N_2 \end{bmatrix} \tag{2.15}$$

As a result of division by Ω_2^2

$$\begin{bmatrix} -\mu_2^2 + 2\mathrm{i}\mu_2\gamma_1\Lambda + \Lambda^2 & -2\mathrm{i}\mu_2\gamma_1\Lambda - \Lambda^2 \\ -V\Lambda(\Lambda + 2\mathrm{i}\mu_2\gamma_1 & \Lambda^2 V + 1 - \mu_2^2 + 2\mathrm{i}\mu_2(\gamma_1 V\Lambda + \gamma_2) \end{bmatrix} \begin{bmatrix} \dfrac{A}{\Omega_2^2} \\ \dfrac{B}{\Omega_2^2} \end{bmatrix} = \begin{bmatrix} 0 \\ N_2 \end{bmatrix} \tag{2.16}$$

Having introduced the following notation

$$\begin{aligned} &\mathbf{W} \begin{bmatrix} \dfrac{A}{\Omega_2^2} \\ \dfrac{B}{\Omega_2^2} \end{bmatrix} = \begin{bmatrix} 0 \\ N_2 \end{bmatrix} \\ &\mathbf{W} = \begin{bmatrix} -\mu_2^2 + 2\mathrm{i}\mu_2\gamma_1\Lambda + \Lambda^2 & -2\mathrm{i}\mu_2\gamma_1\Lambda - \Lambda^2 \\ -V\Lambda(\Lambda + 2\mathrm{i}\mu_2\gamma_1) & a + \mathrm{i}b \end{bmatrix} \end{aligned} \tag{2.17}$$

where

$$a = \Lambda^2 V + 1 - \mu_2^2 \qquad\qquad b = 2\mu_2(\gamma_1 V\Lambda + \gamma_2)$$

the determinant of the matrix $\mathbf{W}$ can be described by the relation

$$|\mathbf{W}| = (-\mu_2^2 + 2\mathrm{i}\mu_2\gamma_1\Lambda + \Lambda^2)(a + \mathrm{i}b) - V\Lambda(\Lambda + 2\mathrm{i}\mu_2\gamma_1)(2\mathrm{i}\mu_2\gamma_1\Lambda + \Lambda^2) \tag{2.18}$$

Equation (2.18) can also be written in the following form

$$|\mathbf{W}| = C_1 + \mathrm{i}D_1 \tag{2.19}$$

where

$$\begin{aligned} C_1 &= \mu_2^4 - \mu_2^2[1 + \Lambda^2(1+V) + 4\gamma_1\gamma_2\Lambda] + \Lambda^2 \\ D_1 &= 2\mu_2\gamma_2\Lambda^2 + \gamma_1\Lambda - \mu_2^2[\gamma_1\Lambda(1+V) + \gamma_2] \end{aligned}$$

The amplitudes of displacements can be computed on the basis of the following relations

$$\begin{aligned} A = x_1\Big|_{max} &= \frac{N_2\Lambda^2}{K_1}\sqrt{\frac{a^2+b^2}{C_1^2+D_1^2}} \\ B = x_2\Big|_{max} &= \frac{N_2}{K_2}\sqrt{\frac{\Lambda^4 + 4\gamma_1^2\Lambda^2\mu_2^2}{C_1^2+D_1^2}} \end{aligned} \tag{2.20}$$

In the further part of the paper, for the purposes of notation simplification, under the terms x_1 and x_2 the amplitudes A and B of displacements will be understood.

In order to verify the correctness of the obtained relations, numerical data was selected in a way making it possible to compare the responses determined on the basis of formulated relations with the example presented in Goliński (1979). For that purpose, the ratios of masses $M_1/M_2 = V = 1$ and frequencies $\Omega_1/\Omega_2 = \Lambda = 4$ were assumed. In Fig. 3 there are presented responses (displacements) of the system characterised by dimensionless damping factors: $\gamma_1 = 0.01$ and $\gamma_2 = 0.001$, stiffness: $K_1 = 16000\,\mathrm{N/m}$, $K_2 = 1000\,\mathrm{N/m}$ and mass $M_1 = 20\,\mathrm{kg}$. The system has two resonant frequencies, corresponding to $\mu_2 = 1/\sqrt{2}$ and $\mu_2 = 5.7$. The minimum of displacement of the mass M_2 corresponds to $\mu_2 = 4.123$.

In Fig. 4, relation between the amplitudes of two-mass system stationary vibrations in function of μ_2 with damping neglected, taken from Goliński (1979) (page 178) is shown.

Comparison of the amplitude-frequency characteristics of displacements x_1 and x_2 of masses M_1 and M_2 determined analytically (Fig. 3) with the

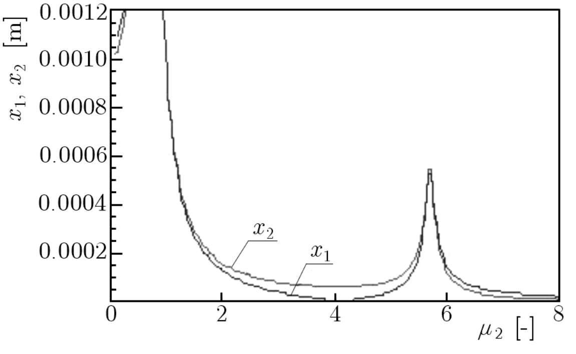

Fig. 3. Displacements x_1 and x_2 of the considered masses presented in functions of the frequency ratio μ_2

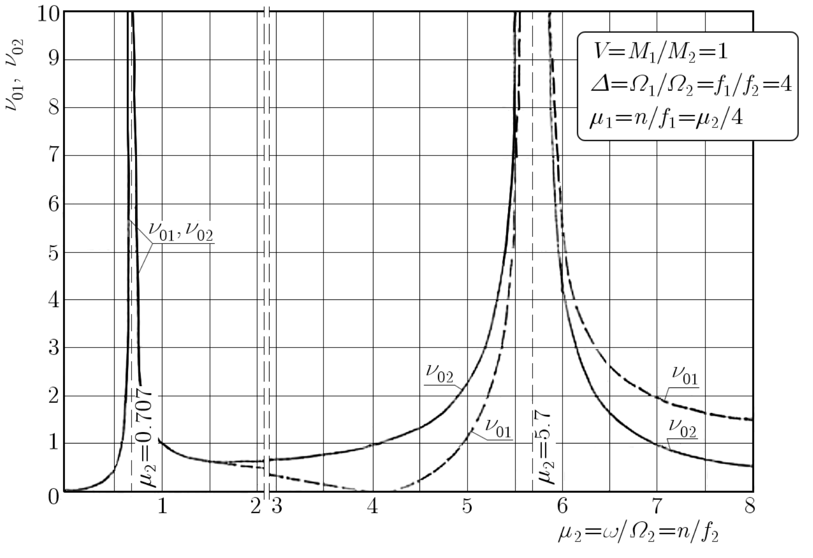

Fig. 4. Amplitude-frequency characteristics of displacements of two-degree-of-freedom system in function of μ_2 (with the influence of damping neglected) (Goliński, 1979)

characteristic presented in Goliński (1979) (see Fig. 4) proved correctness of the considerations presented above.

Perturbation of the considered system boundary conditions by introducing a change in the value of mass M_2 (by ΔM_2) results in a change in the system response. In Fig. 5, the amplitude-frequency characteristic of displacement x_1 of mass M_1 and x_2 of mass M_2 in function of μ_2 and the value of mass $(M_2 + \Delta M_2)$ is presented.

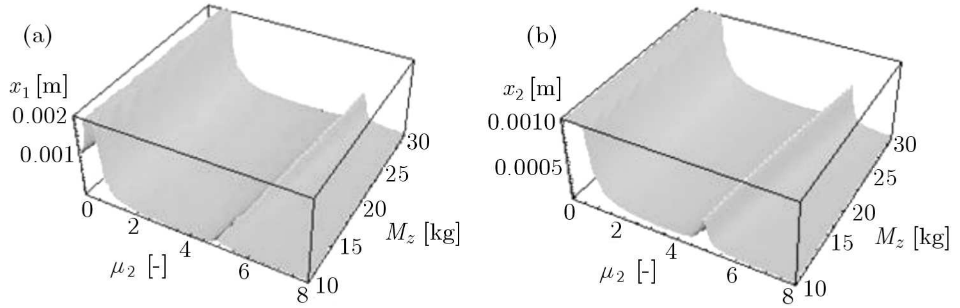

Fig. 5. Amplitude-frequency characteristic of displacements of mass (a) M_1 and (b) M_2 in function of μ_2 and the mass value $(M_2 + \Delta M_2)$, $\Delta M_2 \in \langle -10, +10 \rangle$, where $M_z = M_2 + \Delta M_2\,\mathrm{kg}$

The transfer functions T_{21} and T'_{21}, defined by the ratio of displacements of the system masses, can be computed on the basis of the following relations

$$\begin{aligned} T_{21} &= \frac{x_2}{x_1} = \frac{K_1}{K_2 \Lambda^2}\sqrt{\frac{\Lambda^4 + 4\gamma_1^2 \Lambda^2 \mu_2^2}{a^2 + b^2}} \\ T'_{21} &= \frac{x'_2}{x'_1} = \frac{K_1}{K_2 \Lambda'^2}\sqrt{\frac{\Lambda'^4 + 4\gamma_1^2 \Lambda'^2 \mu_2'^2}{a'^2 + b'^2}} \end{aligned} \tag{2.21}$$

Taking into account equations (2.7), for the considered method, the relations describing T_{21} and T'_{21} are as follows $(k = 1, 2, \ldots, N_f,\ p = 1, 2, \ldots, N'_f)$

$$\begin{aligned} T_{21}(\mathrm{i}\omega) &= \frac{X_2(\mathrm{i}\omega)}{X_1(\mathrm{i}\omega)} \Rightarrow K_2\Big(1 - \frac{1}{T_{21}(\omega_k)}\Big) + K_3 = \omega_k^2 M_2 \\ T_{21}(0) &= \frac{K_2}{K_2 + K_3} \\ T'_{21}(\mathrm{i}\omega) &= \frac{X'_2(\mathrm{i}\omega)}{X'_1(\mathrm{i}\omega)} \Rightarrow K_2\Big(1 - \frac{1}{T'_{21}(\omega_p)}\Big) + K_3 = \omega_p^2 (M_2 + \Delta M_2) \end{aligned} \tag{2.22}$$

hence

$$M_2 = \Delta M_2 \Big(\frac{T_{21}}{T'_{21}} \frac{T'_{21} - T_0}{T_{21} - T_0} - 1\Big)^{-1} = \Delta M W(T'_{21}, T_{21}) \tag{2.23}$$

where

$$\begin{aligned} T_{21} &= T_{21}(\omega_k) \qquad\qquad T'_{21} = T'_{21}(\omega_p) \\ W(T'_{21}, T_{21}) &= \Big(\frac{T_{21}}{T'_{21}} \frac{T'_{21} - T_0}{T_{21} - T_0} - 1\Big)^{-1} \end{aligned}$$

The value of mass M_2 is estimated on the basis of product (2.23) of the value of additional mass ΔM_2 and function $W(T'_{21}, T_{21})$. Function $W(T'_{21}, T_{21})$ is determined experimentally by measurement of the transfer functions T_{21} and T'_{21}.

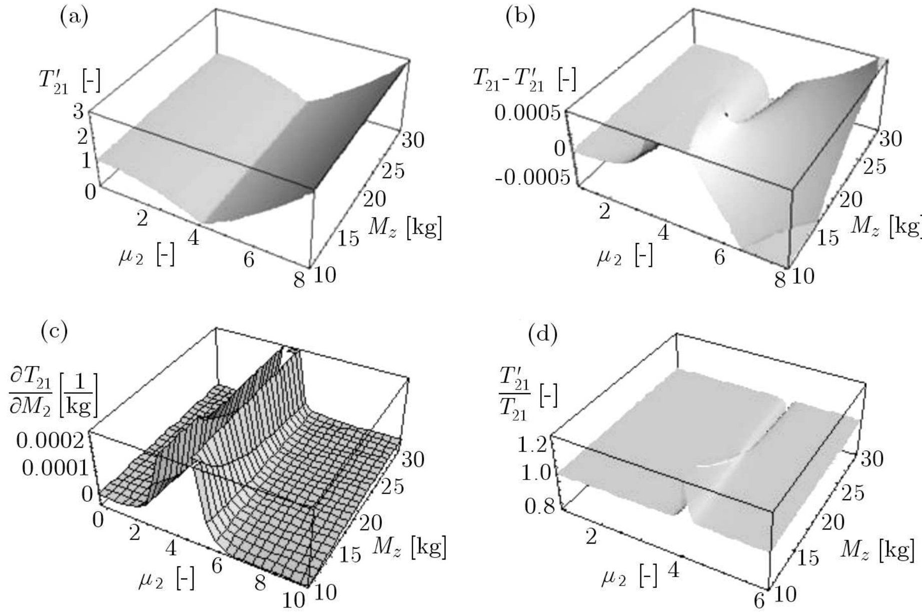

Fig. 6. Amplitude-frequency characteristic of: (a) transfer function T'_{21}, (b) $T'_{21} - T_{21}$, (c) derivative $\partial T_{21}/\partial M_2$, (d) T'_{21}/T_{21} in the function of mass $(M_2 + \Delta M_2)$ and μ_2 (values obtained for $\Delta M_2 \in \langle -10, +10\rangle$), where $M_z = M_2 + \Delta M_2\,\mathrm{kg}$

In Fig. 6, there is presented the transfer function T'_{21} and difference $T'_{21} - T_{21}$ resulting from a change in the value of mass M_2 by $\pm 50\%$. It can be easily noticed that in spite of the significant change in the mass value, changes in the magnitudes of functions T_{21} and T'_{21} are relatively insignificant, especially at a distance from the perpendicular plane crossing the point determining the extremum of the function W (in the considered case $\mu_2 \approx 4.12$). A diagram of the function T'_{21} derivative computed with respect to the mass M_2 is presented in Fig. 7a. Isolines almost parallel to the M_2 axis reveal small sensitivity of the transfer function T'_{21} to changes in the M_2 values, in contrast to much higher sensitivity to changes in μ_2, especially for $\mu_2 \to\sim 4.12$. That sensitivity is one of the sources of system parameter estimation errors and results in the necessity of accurate estimation of thethe transfer functions

T_{21}, T'_{21}, local minimum of the function W (2.23) and careful selection of the value of the additional mass ΔM_2.

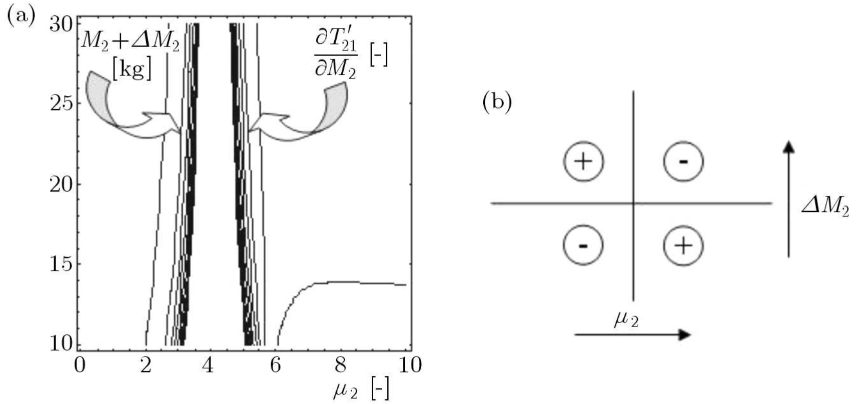

Fig. 7. Sensitivity $\partial T'_{21}/\partial M_2$ of the transfer function to changes in mass M_2 – arrows determine the direction of increase in the derivative (a), monotonicity of the function determining the mass M_2 in the neighborhood of frequency corresponding to the minimum of transfer function and additional mass (b)

Figure 8 depicts the function W plotted in function of μ_2 and the additional mass ΔM_2 determined by a relative percentage change d in the mass value

$$d = \frac{\Delta M_2}{M_2} \cdot 100\% \tag{2.24}$$

It can be noticed that the function $W(T'_{21}(\mu_2 \Delta M_2), T_{21}(\mu_2 \Delta M_2))$ depends heavily on the frequency and additional mass, especially in the vicinity to resonant frequencies. The accuracy of mass M_2 estimation depends on the accuracy of estimation of local extremum of the function W

$$\frac{\partial W(T'_{21}(\mu_2 \Delta M_2), T_{21}(\mu_2 \Delta M_2))}{\partial \mu_2} = 0 \tag{2.25}$$

Roots of equation (2.25), determining the extremum of the function $W(T'_{21}(\mu_2 \Delta M_2), T_{21}(\mu_2 \Delta M_2))$, in the further part of the paper will be referred to as frequency of antiresonance $\mu_2|_{ant-rez}$.

The relation determining the value of mass M_2 (2.23) in the neighborhood of the frequency corresponding to the minimum of transfer function (Fig. 8) is a function increasing or decreasing in four ranges of μ_2 and ΔM_2 (Fig. 7b) antisimetrically with respect to the plane of frequency of the antiresonance $\mu_2|_{ant-rez}$ and nominal mass M_2 ($\Delta M_2 = 0$).

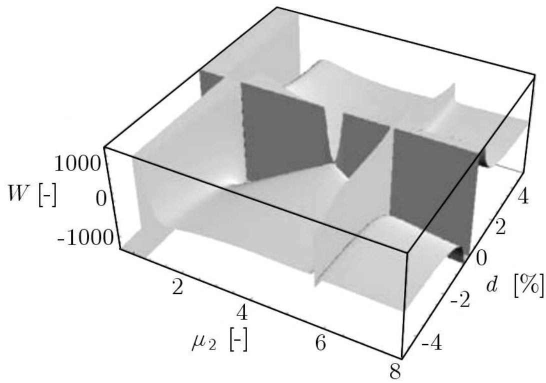

Fig. 8. Diagram of the function $W(T'_{21}(\mu_2\varDelta M_2), T_{21}(\mu_2\varDelta M_2))$ with respect to frequency (determined by μ_2) and relative percentage change d in the additional mass $\varDelta M_2$

In order to avoid searching for the extremum (minimum or maximum) of function (2.23), in the regions presented in Fig. 7b, it is convenient to write an expression describing the mass M_2 as follows

$$M_2 = \min\Big|\varDelta M_2\Big(\frac{T_{21}}{T'_{21}}\frac{T'_{21}-T_0}{T_{21}-T_0} - 1\Big)^{-1}\Big| = \min|\varDelta M_2 W(T'_{21}, T_{21})| \qquad (2.26)$$

Then the minimum is searched for in the whole domain of the function (2.23).

An important drawback of the discussed method of system parameters estimation consists in the requirement of high accuracy of the "antiresonant" frequency $\mu_2|_{ant-rez}$ estimation and the necessity of application of significant values of the additional mass $\varDelta M_2$ that influences the accuracy of the mass M_2 estimation. Taking into account relation (2.23), the relative error of mass M_2 estimation can be calculated as the sum of errors of the function $W(T'_{21}, T_{21})$ estimation and the additional mass $\varDelta M_2$ determination

$$\delta(M_2) = \delta(\varDelta M_2) + \delta(W(T'_{21}(\mu_2, \varDelta M_2), T_{21}(\mu_2, \varDelta M_2)) \qquad (2.27)$$

The mass $\varDelta M_2$ can be determined with an arbitrary technical accuracy and, therefore, the error of that mass estimation is negligible. Therefore, this error equals the error of estimation of the function $W(T'_{21}, T_{21})$. The measured functions T'_{21}, T_{21} (found theoretically by relations $(2.5)_1$ and (2.6)) are experimentally determined complex functions depending on μ_2 and $\varDelta M_2$. In the further part of the paper, the influence of the estimation accuracy of the "antiresonant" frequency $\mu_2|_{ant-rez}$ and the selection of additional mass $\varDelta M_2$ on the accuracy of mass M_2 estimation will be discussed.

In order to determine the influence of accuracy of the "antiresonant" frequency $\mu_2|_{ant-rez}$ estimation and the selection of the additional mass ΔM_2 value on the variance of mass M_2, relations (2.8) and (2.12) were used. Matrix $\mathbf{T}$ is as follows

$$\mathbf{T} = \left[\frac{\partial W(T_{21}(\mu_2, \Delta M_2), T'_{21}(\mu_2, \Delta M_2)}{\partial \mu_2} \quad \frac{\partial W(T_{21}(\mu_2, \Delta M_2), T'_{21}(\mu_2, \Delta M_2))}{\partial \Delta M_2}\right] \tag{2.28}$$

Consecutive elements of matrix $\mathbf{T}$, $\partial W/\partial \mu_2$ and $\partial W/\partial \Delta M_2$ are presented in Fig. 9.

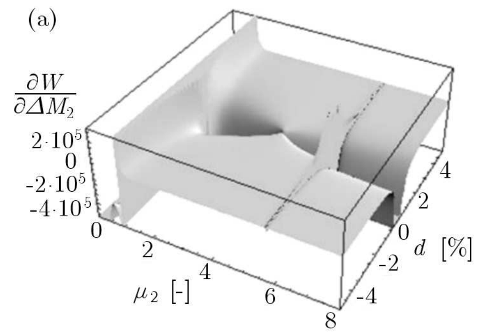

Fig. 9. Elements $\partial W/\partial \Delta M_2$ and $\partial W/\partial \mu_2$ of the matrix $\mathbf{T}$

In Fig. 10, a diagram of the mass M_2 variance is presented. Isometric projection on the plane d, μ_2 makes it possible to determine the course of lines of the same value of variance. The minimum of the mass M_2 variance lies in the vertical plane containing the straight line $\mu_2 = \mu_2|_{ant-rez}$.

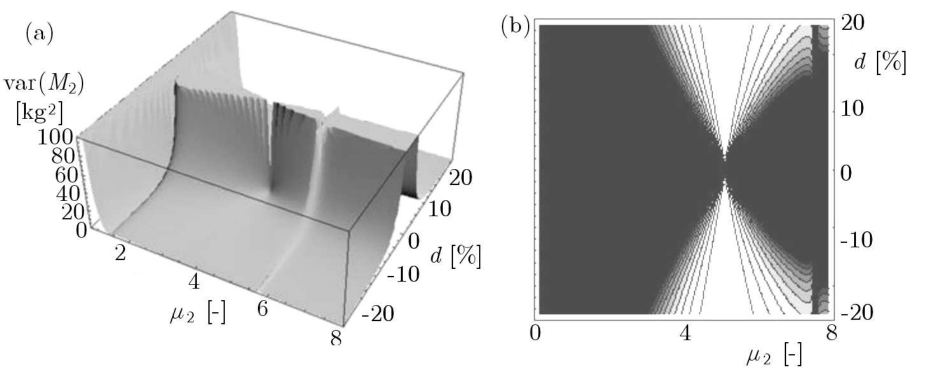

Fig. 10. Variance of mass M_2 in function of d and μ_2, (a) three-dimensional function shape, (b) isometric projection on the plane d, μ_2

The variance of mass M_2, described by the relation:

$$\begin{aligned}\mathrm{var}(M_2) = \sum_{j=1}^{n}\Big(\frac{\partial M_2(T_{21}(\mu_2,\varDelta M_2),T'_{21}(\mu_2,\varDelta M_2)}{\partial\mu_2}\Big)^2\Big|_{\mu_2=\mu_2|_{ant-rez}}\sigma^2(\mu_2) + \\ + \sum_{j=1}^{n}\Big(\frac{\partial M_2(T_{21}(\mu_2,\varDelta M_2),T'_{21}(\mu_2,\varDelta M_2)}{\partial\varDelta M_2}\Big)^2\Big|_{\varDelta M_2=0}\sigma^2(\varDelta M_2) + \ldots\end{aligned} \tag{2.29}$$

makes it possible to analyse the overall influence of estimation accuracy of many input quantities on the total relative (percentage) error of the mass M_2 estimation

$$\delta_{\%}(M_2) = \sqrt{\frac{E[(M_2-\widehat{M}_2)^2]}{M_2^2}} = \frac{\sqrt{\mathrm{var}(M_2)}}{M_2}\cdot 100\% \tag{2.30}$$

In Fig. 11 the characteristic of relative error (2.30) of the mass M_2 estimated on the basis of change in the system response resulting from introduction of an additional mass $\varDelta M_2(\pm 25\% M2)$ is presented. It was assumed that the mass $\varDelta M_2$ is determined with the accuracy of 1%, while the accuracy of the "antiresonance" frequency $\mu_2|_{ant-rez}$ estimation equals 0%, 1%, 2%, 3%, 5% and 7%. In Fig. 11, the same relation plotted for the change in the mass value by $\pm 5\%$ is presented.

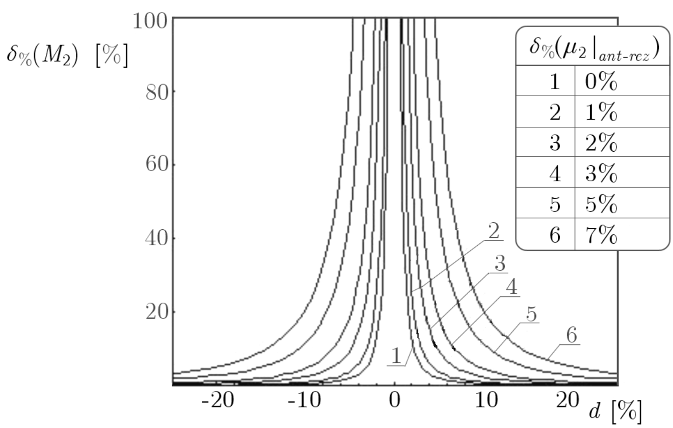

Fig. 11. Relative error of the mass M_2 estimated on the basis of change in the system response resulting from addition or subtraction of the mass $\varDelta M_2$ in the range of $d = \pm 25\%$

On the basis of Fig. 11, it can be stated that the relative error of the mass M_2 estimation, for a given accuracy of the "antiresonant" frequency $\delta_{\%}(\mu_2|_{ant-rez})$ estimation, depends exponentially on the mass change

$d = (\Delta M_2/M_2) \cdot 100\%$ and approaches infinity for $d \to 0$. The curve drawn for $\delta_{\%}(\mu_2|_{ant-rez}) = 0$ determines the method error. Such results are consistent with the results presented in Fig. 10. In order to obtain the required accuracy of the mass M_2 estimation, it is necessary to increase the value of ΔM_2 for a constant value of $\delta_{\%}(\mu_2|_{ant-rez})$ or, for the assumed value of ΔM_2 kg or d [%] to increase the accuracy of "antiresonant" frequency $\delta_{\%}(\mu_2|_{ant-rez})$ estimation. Theoretically, the measurements of frequency can be carried out with a high accuracy, nevertheless, in practice, the estimation of the "antiresonant" frequency $(\mu_2|_{ant-rez})$ on the basis of the previously determined extremum of the transfer function T_{21} and T'_{21} requires selection of the proper excitation, e.g. of the swept-sine type. When introduction of the further changes to the mass ΔM_2 is not possible, the accuracy of μ_2 estimation should be increased, taking into account longer time of measurements and the necessity of careful selection of the excitation type.

3. Conclusions and final remarks

The paper concerns the operational identification method that makes it possible to estimate parameters of linear as well as nonlinear mechanical systems on the basis of system dynamic responses measured during the normal system work. The algorithm of the method consists in sequential application of the restoring force, boundary perturbations and direct parameter identification techniques. The obtained parameter estimates provide information concerning forces transferred to the foundation that can be used for the purposes of early damage detection and, what follows, minimization of negative influence of vibrations transferred to the foundations and the environment. Information from the monitoring process can be treated as a basis for making a decision on further machine exploitation and the input data to the system of automatic control, which would control braking in order to minimize potential threat.

In the paper, there are also presented the results of the sensitivity analysis carried out in order to assess the influence of input parameters uncertainties (accuracy of resonant frequency and amplitude estimates, errors of transfer function estimation in operational conditions, value of the introduced additional mass) on the accuracy of estimated system parameters. The presented analysis makes it possible not only to determine the sensitivity of estimated system parameters, such as mass M_2 or stiffness coefficient K_2, to measurement errors of the input variables but also to select the size of introduced structural modification (change in the mass M_2) minimizing the error of estimated parameters for the given accuracy of measurements of input variables.

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Analiza wrażliwości metody identyfikacji układów nieliniowych w warunkach eksploatacyjnych

Streszczenie

W pracy przedstawiono metodę operacyjnej identyfikacji parametrów modeli nieliniowych konstrukcji mechanicznych, realizowaną w oparciu o algorytmy metody sił resztkowych, zaburzeń brzegowych oraz bezpośredniej identyfikacji parametrów. Uzyskane estymaty parametrów dostarczają informacji o siłach przekazywanych na podłoże i znajdują zastosowanie w diagnostyce realizowanej w oparciu o model układu nieuszkodzonego.

W celu oszacowania wpływu niepewności parametrów wejściowych na dokładność estymowanych parametrów układu, przeprowadzono analizę wrażliwości poszukiwanych parametrów układu (masy, sztywności, tłumienia) na dokładność estymacji częstotliwości i amplitud rezonansowych (uwarunkowaną błędami estymacji funkcji przejścia w warunkach eksploatacyjnych), a także wartość masy wprowadzanej do układu w celu zmodyfikowania jego własności dynamicznych (zgodnie z algorytmem metody zaburzeń brzegowych).

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