JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
50, 1, pp. 231-250, Warsaw 2012
50th anniversary of JTAM

PIECEWISE RELIABILITY-DEPENDENT HAZARD RATE FOR COMPOSITES UNDER FATIGUE LOADING ADJUSTMENT

Chung-Ling Chen
Kuo-Shong Wang

National Central University, Department of Mechanical Engineering, Jhongli, Taiwan, Republic of China; e-mail: sysengrg@hotmail.com.tw

Based on the derived transition period and reliability drop, this paper proposes a method of piecewise combination of the reliability-dependent hazard rate function named (e_ocp) model to describe the dynamical reliability in a two-stage fatigue loading process. First, the parameters e_o, c, p are fitted through simulated failure data under various constant-amplitude cyclic stresses. The reliability of the high-low loading process is described piecewise with the corresponding values of (e_o, c, p) for each respective stress level, and maintains R_a in the transition period while R_a denotes the reliability at which the stress level changes. The reliability of the low-high process is determined by subtracting the portion of reliability drop at R_a from the piecewise fitted curves. The proposed reliability behavior is verified successfully. The linear damage sum is found to be larger than unity for the high-low loading, and on the contrary for the low-high cases. A larger difference between the stress level changed results in larger deviation of damage sum from unity, especially when R_a near 0.9.

Key words: fatigue loading adjustment, hazard rate function, dynamical reliability, Monte Carlo simulation, linear damage sum

1. Introduction

The dynamical reliability of composite laminates when subjected to fatigue loading adjustment is a fundamental issue in evaluating these materials for practical applications. Several researchers (Broutman and Sahu, 1972; Yang and Jones, 1980, 1981, 1983; Gamstedt and Sjögren, 2002; Found and Quaresimin, 2003) have reported that when composites are no longer able to sustain the fatigue load, Miner's damage sum will be larger than unity in the high-low sequence and smaller than unity in the low-high sequence. In contrast, others (Han and Hamdi, 1983; Hwang and Han, 1986) have reached the opposite conclusion for other types of constituent materials. Regardless, little attention has been focused on an explanation of the load sequence effect based on the dynamical reliability of composites under varied stress-level fatigue situations.

As for the dynamical reliability of materials subjected to two-stage cyclic stresses, only limited research has been done successfully in this area. This is mainly because the sample size of most two-stage fatigue tests is too small to verify statistical analysis accurately. Tanaka *et al.* (1984) used the B-model to analyze the probability distribution of fatigue life of a large size of nickel-silver samples. However, it is difficult to apply this model to predict the behavior in a two-stage loading process when only results of a single-stage fatigue test are available. After the development of several hazard rate models as reviewed by Wang (2011), a two-parameter reliability-dependent hazard rate function $h(R) = e_o + c(1 - R)$ is used to deal with the dynamical reliability of a material concerning fatigue loading adjustment (Wang *et al.*, 1997). When the stress level of fatigue loading is adjusted, the hazard rate right before the adjustment becomes the intrinsic weakness at the beginning of the following stage loading. This relation has been verified by the data given by Tanaka *et al.* (1984). Later, Ni and Zhang (2000) presented a two-stage fatigue reliability method based on two-dimensional probabilistic Miner's rule. The results are also verified by the data of Tanaka *et al.* (1984), but the application of this method is restricted by some assumptions. The composites are inhomogeneous and anisotropic materials, and more complicated in the fatigue behavior and failure mechanisms than those of homogeneous and isotropic metallic materials. The above methods have not proven to be valid for composites yet. Wang *et al.* (2002) modified the above two-parameter hazard rate relation to a three-parameter form of $h(R) = e_o + c(1 - R)^p$, the so-called $(e_o cp)$ model, to depict the dynamical reliability of several types of engineering components and devices. This model has been verified to describe the dynamical reliability of composite laminates under simulated single-stage fatigue loading with good results (Chen *et al.*, 2009). In the region of high cycle fatigue of composites, it is found that e_o and p can be considered as a fixed value; c can be a power function of the stress level.

Recently, Chen and Wang (2011) defined two parameters, the transition period n_{2a} and reliability drop $|\Delta R|$ (see Appendix), respectively, to describe the effect of high-low and low-high fatigue loading adjustment on the reliability degradation of composite materials. Figure 1 shows a typical expression of the reliability degradation of composite laminates under two-stage fatigue loading processes. Denote the reliability at the instant of loading adjustment by R_a. In the high stress section of both the high-low and low-high loading processes, the strength of composite laminates degrades at a relatively higher speed than that under a low level stress. Consequently, the higher rate of fatigue failure causes the reliability to degrade relatively steeply. At the instant the stress is adjusted from high to low level, the residual strength of the survivals becomes larger than the low-level maximum cyclic stress. During a period of n_{2a}, named the transition period, no failure occurs until the minimum residual strength degrades to the low-level maximum cyclic stress. Thus, the reliability remains unchanged in n_{2a}. Analogously, at the instant of low-high adjustment, those specimens with a residual strength with magnitude between the two levels fail right away and the reliability drops sharply by $|\Delta R|$.

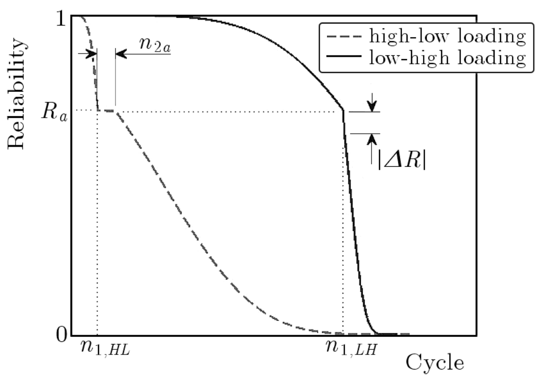

Fig. 1. Typical expression of reliability degradation of composites under two-stage loading

The purpose of this paper is to extend the application of the $(e_o cp)$ model for single-stage fatigue loading to two-stage cases, using a piecewise combination with n_{2a} or $|\Delta R|$ to describe the whole picture of dynamical reliability. The reliability in the high-low loading process can be divided into three sections: a high stress section, a transition period, and a low stress section. A modification equation of the parameter c for the low stress section of the high-low loading is proposed to get better fitting of the model with the fatigue failure data. In the low-high case, it initially follows the behavior of low-level stress situation until the stress adjusting, then with a simultaneous drop $|\Delta R|$, it degrades as the case at high-level stress afterwards. Miner's rule provides a simple way to predict the fatigue life of materials under a staged fatigue loading; nevertheless, it does not address the effect of the load sequence on the fatigue life of the composite. Here, based on the dynamical reliability, we present a way to estimate the linear damage sum in large populations of composites under various two-stage fatigue loading processes. The present study is the first to describe accurately the dynamical reliability of composites under a two-stage fatigue loading and explains the effect of stress level, instant of adjustment and load sequence on the linear damage sum of composite materials.

2. Piecewise hazard rate function and linear damage sum

2.1. $(e_o cp)$ Model

By definition, the hazard rate $h(t)$ is related to the reliability $R(t)$ as follows

$$h(t) = -\frac{1}{R(t)}\frac{dR}{dt} \tag{2.1}$$

In a deteriorating system, the reliability $R(t)$ degrades monotonically with time t, thus R corresponds to t in a one-to-one relationship. This leads the time-dependent hazard rate function $h(t)$ to be expressed in terms of reliability R as $h(R)$. Wang *et al.* (2002) proposed a reliability-dependent hazard rate function, named the $(e_o cp)$ model, in the form of

$$h(R) = e_o + c(1-R)^p \qquad e_o > 0, \quad c > 0, \quad p > 0 \tag{2.2}$$

where e_o is defined as the imbedded decay factor which takes account of the intrinsic defects during the manufacturing of the mechanical elements. The parameter c represents the process-dependent decay factor which is concerned with the rate of damage accumulation of materials under loading. A larger value of c represents a larger hazard rate resulting from the higher fatigue stress level or other types of heavier mechanical loading. The parameter p denotes the beginning of noticeable degradation in reliability, referring to the memory characteristic of the damage. Assume the static strength of composite materials to have a two-parameter Weibull distribution, as in the widely accepted cases. When the composites are subjected to a constant-amplitude maximum cyclic stress S at a certain stress ratio and a certain frequency, the corresponding values of (e_o, c, p) can be obtained by fitting Eq. (2.2) with the fatigue failure data. It is found that in the region of high cycle fatigue of composites that e_o and p can be taken to have a fixed value while c is correlated as a power relation for the ratio S/β as follows (Chen *et al.*, 2009)

$$c = \varepsilon\Big(\frac{S}{\beta}\Big)^{\lambda} \tag{2.3}$$

where β is the scale parameter of the Wiebull static strength distribution; ε and λ are related to the initial material characteristics.

To express the reliability of a composite under constant-amplitude cyclic stress as a function of fatigue cycles n, $R(n)$, the mean cycles to failure (MCTF) of composite specimens can be calculated by integrating $R(n)$

$$\overline{N} = \int\limits_0^\infty R(n)\; dn \tag{2.4}$$

Replacing t with n and substituting Eq. (2.1) into Eq. (2.4) allows $\overline{N}$ to be

$$\overline{N} = -\int\limits_1^0 \frac{1}{h}\; dR = -\int\limits_1^0 \frac{1}{e_o + c(1-R)^p}\; dR \tag{2.5}$$

Let $1 - R = F$, $-dR = dF$. It leads the above integration to be

$$\begin{aligned} \overline{N} &= \int\limits_0^1 \frac{1}{e_o + cF^p}\; dF = \frac{1}{e_o}\int\limits_0^1 \frac{1}{1+\frac{c}{e_o}F^p}\; dF = \frac{1}{e_o}\int\limits_0^1 \sum_{k=0}^{\infty}\Big(\frac{c}{e_o}F^p\Big)^k\; dF \\ &= \frac{1}{e_o}\sum_{k=0}^{\infty}\int\limits_0^1 \Big(\frac{c}{e_o}\Big)^k F^{pk}\; dF = \frac{1}{e_o}\sum_{k=0}^{\infty}\Big(\frac{c}{e_o}\Big)^k \frac{1}{pk+1} + C_i \end{aligned} \tag{2.6}$$

where C_i is the constant of integration. To save the work of integrating the above equation, an approximated equation of fatigue life (Shih, 2000) is proposed in terms of c/e_o as

$$c\mu = \rho_1\Big(\frac{c}{e_o}\Big)^{\upsilon_1} + \rho_2\Big(\frac{c}{e_o}\Big)^{\upsilon_2} \tag{2.7}$$

where μ is the approximated mean fatigue life of the composite under the maximum cyclic stress S; the other parameters ρ_1, ρ_2, υ_1 and υ_2 are given in tables.

2.2. Modification of parameter c in high-low loading

Consider a two-stage fatigue loading process in composite materials, where S_1 represents the first stage maximum cyclic stress, and S_2 the second stage. Let the reliability at the instant of load adjusting be R_a. Denote $\overline{e_o}$, c_1, $\overline{p}$ as the parameters fitted in the (e_ocp) model for S_1, and $\overline{e_o}$, c_2, $\overline{p}$ for S_2. Thus the variation of the hazard rate under various stress levels can be mainly determined by the ratio S/β which appears in the representation of c, as shown in Eq. (2.3).

In the high stress section of the high-low loading process, the residual strength of the survivals will degrade at the same rate as in a single-stage loading process at high-level stress, in other words, the same as the reliability does. In this section, the process-dependent decay factor c_1 is decided by the high-level stress S_1. Right after the high-low adjustment, the reliability remains at R_a during the transition period n_{2a}. After the transition period, c_2 is basically decided by the low-level stress S_2. However, the survivals after the high-stress section and the transition period should have experienced more cumulative damage than those specimens under a single-stage loading at low-level stress. Thus the residual strength will degrade further after the transition period. As a matter of fact, c_2 is replaced by c_2' as in

$$c_2' = \eta(n_{2a}, S_1, S_2)c_2 \tag{2.8}$$

where c_2 is given for a single-stage loading at low-level stress S_2, η is a function of S_1, S_2 and n_{2a} for modifying c_2 in the low stress section of a high--low loading process. The modification for c_2 indicates the hidden degradation which exists in composites under the load S_2 in the free-failure period n_{2a}. Thus, η should be larger than unity; a longer n_{2a} implies a larger η. It can be seen in Eqs. (A.1)-(A.4), for certain composite laminates with specific values of α, β, K, b, ω, d and α_f, n_{2a} is a function of S_1, S_2 and R_a. For fixed values of S_1 and S_2, n_{2a} increases monotonically with the decreasing R_a, thus c_2' can be further reduced to

$$c_2' = \eta(R_a)c_2 \tag{2.9}$$

To obtain a better fit for the low stress section of a high-low loading process, $\eta(R_a)$ is proposed to modify c_2 as in

$$\eta(R_a) = 1 + \zeta\Big(\frac{1-R_a}{R_a}\Big)^{\gamma} \tag{2.10}$$

where ζ and γ are related to the material characteristics of composites.

2.3. Piecewise combination of hazard rate function

For the low-high situation, the reliability in the first section $(R \geqslant R_a)$ is described by the hazard rate with $(\overline{e_o}, \overline{p}, c_1)$ under low-level stress conditions. The moment the stress level is increased from S_1 to S_2, failure occurs right away in those survival specimens of which the residual strengths are between S_1 and S_2 in magnitude. Thus, the reliability instantly drops by $|\varDelta R|$ (see Eqs.

(A.5)-(A.7)). The remaining specimens after the reliability drop are considered to have experienced nearly the cumulative damage as those specimens having experienced a single-stage process at high-level stress. Thus, the reliability after the reliability drop follows the hazard rate as described by $(\overline{e_o}, \overline{p}, c_2)$ for a single-stage under high-level stress. Thus, the hazard rate in the first stage is

$$h_1 = \overline{e_o} + c_1(1-R)^{\overline{p}} \qquad 1 > R > R_a \tag{2.11}$$

and in the second stage it is

$$h_2 = \overline{e_o} + c_2(1-R)^{\overline{p}} \qquad R < R_a \tag{2.12}$$

where the values of c_1 and c_2 are basically decided by Eq. (2.3). The parameter c_2 needs modification as expressed in Eqs. (2.9) and (2.10) to obtain a better fitting in the low stress section of the high-low loading process. Now express the hazard rate in the whole range with a unit step function as

$$h(R) = h_1 u(R-R_a) + h_2[u(R) - u(R-R_a)] \tag{2.13}$$

where $u(R)$ and $u(R-R_a)$ are defined as

$$u(R) = \begin{cases} 0 & \text{for} \quad R < 0 \\ 1 & \text{for} \quad R \geqslant 0 \end{cases} \qquad u(R-R_a) = \begin{cases} 0 & \text{for} \quad R < R_a \\ 1 & \text{for} \quad R \geqslant R_a \end{cases} \tag{2.14}$$

2.4. Mean fatigue cycle and linear damage sum

(a) For the high-low loading process, $S_1 > S_2$. As can be seen in Fig. 1, the mean fatigue cycle for the process includes three parts: for the high stress section

$$\overline{n_1}_{,HL} = \int\limits_0^{n_{1,HL}} R(n)\, dn \tag{2.15}$$

where $n_{1,HL}$ is the number of applied cycles in the first stage of the high-low loading process; for the transition period

$$\overline{n_{2a}}_{,HL} = R_a n_{2a} \tag{2.16}$$

The mean fatigue cycle of low-level stress loading after the transition period is

$$\overline{n_{2b}}_{,HL} = \int\limits_{n_{1,HL}+n_{2a}}^{\infty} R(n)\, dn \tag{2.17}$$

The total mean fatigue cycles of the complete process becomes $\overline{n_1}_{,HL} + \overline{n_{2a}}_{,HL} + \overline{n_{2b}}_{,HL}$.

(b) For the low-high loading process, $S_1 < S_2$. As shown in Fig. 1, the mean fatigue cycle of the process includes two parts. The mean fatigue cycles of the first part is

$$\overline{n_1}_{,LH} = \int\limits_0^{n_{1,LH}} R(n)\, dn \tag{2.18}$$

where $n_{1,LH}$ is the number of applied cycles in the first stage of the low-high loading process. Right after the low-high adjustment, the reliability drops by $|\Delta R|$. In the second part we have

$$\overline{n_2}_{,LH} = \int\limits_{n_{1,LH}}^{\infty} R'(n)\, dn \tag{2.19}$$

where $R'(n)$ is the part of $R(n)$ in the range of $(R_a - |\Delta R|, 0)$. The total mean fatigue cycles of the low-high loading process is $\overline{n_1}_{,LH} + \overline{n_2}_{,LH}$.

(c) For composites in a two-stage fatigue loading process, the linear damage sum is

$$D_m = \frac{\overline{n_1}}{\overline{N_1}} + \frac{\overline{n_2}}{\overline{N_2}} \tag{2.20}$$

where $\overline{n_1}$ and $\overline{n_2}$ are the mean fatigue cycles for the periods under the stress levels S_1 and S_2, respectively; $\overline{N_1}$ and $\overline{N_2}$ are the corresponding mean cycles to failure. Substituting Eqs. (2.15)-(2.17) into Eq. (2.20) yields the linear damage sum for the high-low loading process

$$D_{HL} = \frac{\overline{n_1}_{,HL}}{\overline{N_1}} + \frac{\overline{n_{2b}}_{,HL}}{\overline{N_2}} + \frac{\overline{n_{2a}}_{,HL}}{\overline{N_2}} \tag{2.21}$$

According to Miner's rule, the sum of the first two terms becomes unity; the third term yields the total sum that is larger than unity. Similarly, the linear damage sum for the low-high loading process is

$$D_{LH} = \frac{\overline{n_1}_{,LH}}{\overline{N_1}} + \frac{\overline{n_2}_{,LH}}{\overline{N_2}} \tag{2.22}$$

where $\overline{n_2}_{,LH}$, Eq. (2.19), is smaller than the integral $\int_{n_{1,LH}}^{\infty} R(n)\, dn$ due to the existence of a drop in the reliability. Thus, Miner's damage sum for this case is smaller than unity.

3. Curve fitting with failure data in simulation

Based on the residual strength equations by Yang and Jones (1980, 1981, 1983), this study uses MATLAB package to carry out Monte Carlo simulations of the residual strength degradation and fatigue failure for ISO standard $[\pm 45]_S$ glass/epoxy laminates under single-stage and two-stage loading. There are 16 loading cases as shown in Table 1.

Table 1. Cases of Monte Carlo fatigue loading simulation for G1/Ep$[\pm 45]_S$ laminate

Case No.	Constant--amplitude S	Case No.	High-to-low		Case No.	Low-to-high	
			S_1	S_2		S_1	S_2
1	75.5	7	75.5	56.6	12	56.6	75.5
2	70.8	8	70.8	56.6	13	56.6	70.8
3	66.6	9	66.6	56.6	14	56.6	66.6
4	62.9	10	62.9	56.6	15	56.6	62.9
5	59.6	11	59.6	56.6	16	56.6	59.6
6	56.6						

units: MPa

The stress ratio of cyclic loading is set to be 0.1 for various stress levels. The loading frequency is assumed to be proportional to $1/S^2$ so that overheating of the specimens is avoided. The associated parameters used in the simulations are $\alpha = 59.8$, $\beta = 113.26$, $K = 1.2\text{E-}25$, $b = 11.1806$, $\omega = 4.9633$ and $r = 12.9238$ (Philippidis and Passipoularidis, 2007). The values of parameters (e_o, c, p) for a single-stage fatigue loading under $S = 75.5$, 56.6, 45.3 and 37.8 MPa (i.e., the ratios $\beta/S = 1.5$, 2.0, 2.5 and 3.0), respectively, are obtained in Chen *et al.* (2009). The specific parameter values are $\overline{e_o} = 1\text{E-}12$ and $\overline{p} = 0.84$. Also, the parameters in Eq. (2.3) are $\varepsilon = 0.079246$ and $\lambda = 11.378$. Since the range of the maximum cyclic stress $S = 75.5$-56.6 MPa considered in this paper is within the range $S = 75.5$-37.8 MPa considered in the previous paper of the authors, thus the values of $\overline{e_o}$, $\overline{p}$, ε and λ are the same as above. The simulation procedure of strength degradation and reliability decay in each two-stage fatigue loading case is:

(1) Generate randomly a total of 10^4 samples with the static strengths having a two-parameter Weibull distribution.

(2) Compare each sample strength with the maximum cyclic stress S. The specimens with strength $> S$ are deemed as survivals, and the others as

failures. The value of S is fixed in each stage loading process. The value of S is adjusted at the specified loading cycles (or specified reliability).

(3) Calculate the reliability and hazard rate of composite versus the number loading cycles according to the associated definition in engineering.

(4) Calculate the residual strength $X_S(n)$ of the survivals individually by Eq. (A.4) after each time of simulation with specified additional loading cycles. Repeat the steps (2)-(4) until all specimens fail.

4. Results and discussion

It can be seen in Fig. 2 that the fitted curves of the $(e_o cp)$ model correspond to the simulated data for stress at 75.5, 70.8, 66.6, 62.9, 59.6 and 56.6 MPa, respectively. The fitted values of (e_o, c, p) and MCTF under these stress levels are summarized in Table 2. e_0 and p remain unchanged and c increases with decreasing β/S.

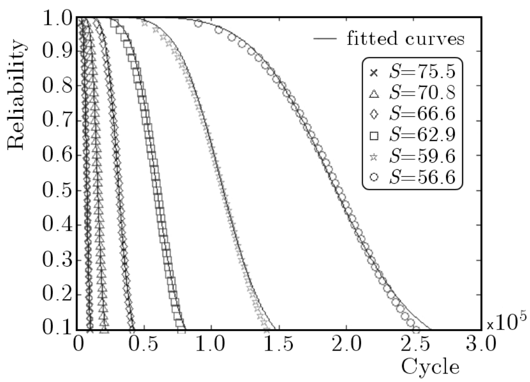

Fig. 2. Curve fitting of the $(e_o cp)$ model for simulated fatigue data for G1/Ep$[\pm 45]_S$ laminate under various constant-amplitude maximum cyclic stresses S

Figure 3 shows that the comparison between the predicted mean fatigue cycles in the transition period $\overline{n_{2a}}_{,HL}$ and the simulated data under various high-low loading conditions is satisfactory. As shown in Eq. (A.1), for fixed values of S_2 and R_a, the larger S_1 the larger value of n_{2a}. For fixed values of S_1 and S_2, n_{2a} increases monotonically with the decrease of R_a to a finite value. Thus, as shown in Fig. 3, $\overline{n_{2a}}_{,HL}$, the product of n_{2a} and R_a, increases steeply at the beginning, and quickly approaches a peak near $R_a = 0.9$, then decreases gradually afterwards.

Table 2. Fitted e_o, p, c and mean cycles to failure for G1/Ep[±45]$_S$ laminates under various stress conditions

S [MPa]	β/S	e_o	p	c	$\overline{N}$ by (2.4) [cycle]
75.5	1.5	1E-12	0.84	7.90E-4	7742
70.8	1.6	1E-12	0.84	3.78E-4	15984
66.6	1.7	1E-12	0.84	1.88E-4	31896
62.9	1.8	1E-12	0.84	9.83E-5	60797
59.6	1.9	1E-12	0.84	5.33E-5	1.1029E+5
56.6	2.0	1E-12	0.84	2.94E-5	1.9627E+5

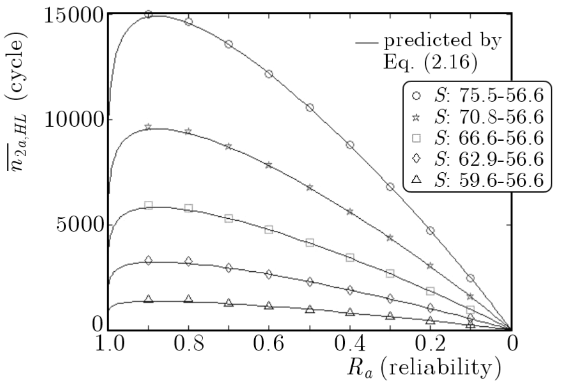

Fig. 3. Comparison between the predicted mean fatigue cycle in the transition period and the simulation data under various high-low fatigue loading adjustments

Figure 4 shows that the typical piecewise fitting of the (e_ocp) model for the simulated data for the hazard rate versus reliability in a high-low process, adjusted from $S_1 = 66.6$ MPa to $S_2 = 56.6$ MPa at $R_a = 0.5$ is satisfactory. It is evident that the hazard rate rises at a relatively higher rate in the high stress section and drops suddenly to zero at the instant of high-low adjustment, $R_a = 0.5$. After the transition period, the reliability degrades from 0.5 and hazard rate continues to increase from a value lesser than that right before $R_a = 0.5$. The slop of the hazard rate appears lesser in the low stress section than in the high stress section.

Figure 5 depicts the step-by-step piecewise fitting of the reliability for the corresponding conditions in Fig. 4. Figure 5a shows the fitted curves under single-stage $S = 66.6$ MPa, where the shaded area represents the mean fatigue cycles $\overline{n_{1,HL}}$ in $1 \geqslant R > 0.5$. Figure 5b shows the fitted result under single-stage $S = 56.6$ MPa, where the shaded area indicates the mean fatigue cycles

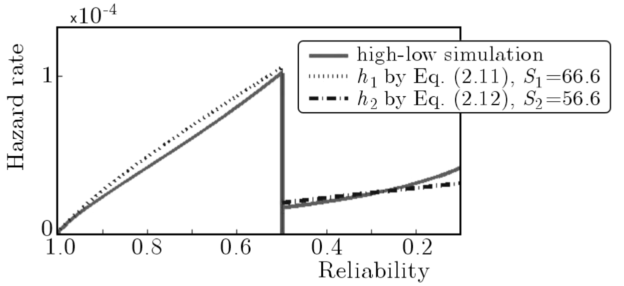

Fig. 4. Typical piecewise fitting of the $(e_o cp)$ hazard rate model for simulated fatigue data for G1/Ep$[\pm 45]_S$ laminate under high-low loading conditions, from $S_1 = 66.6\,\mathrm{MPa}$ to $S_2 = 56.6\,\mathrm{MPa}$ at $R_a = 0.5$

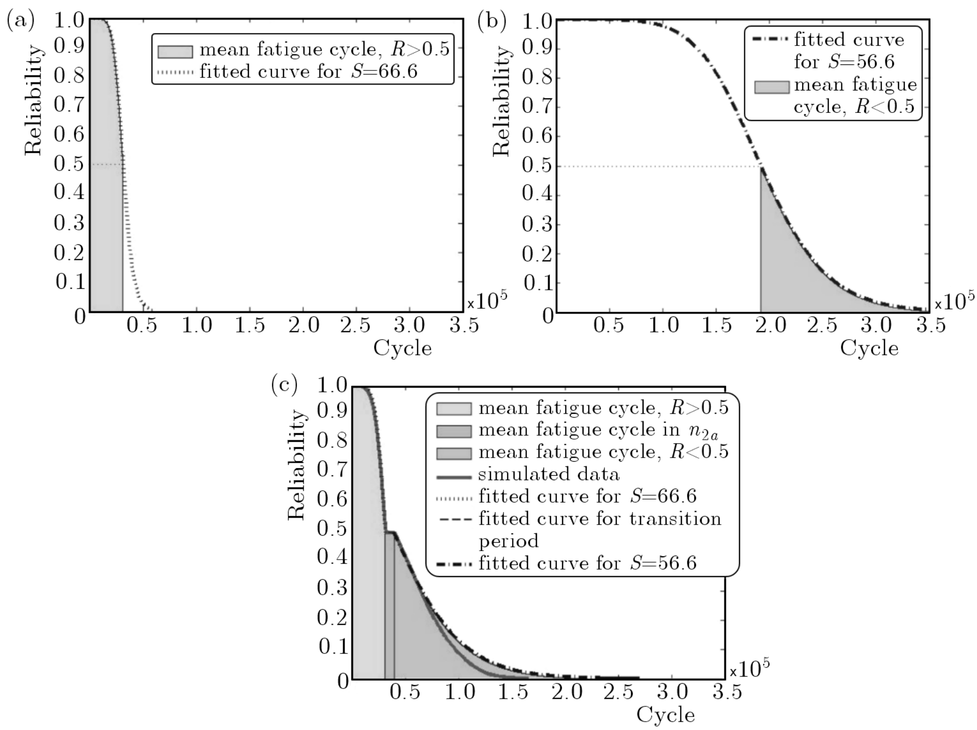

Fig. 5. Typical piecewise fitting of the $(e_o cp)$ model for high-low simulation data for G1/Ep$[\pm 45]_S$: (a) under $S = 66.6$; (b) under $S = 56.6$; (c) adjusted from $S_1 = 66.6\,\mathrm{MPa}$ to $S_2 = 56.6\,\mathrm{MPa}$ at $R_a = 0.5$, with $c_2 = 2.94\text{E-}5$

$\overline{n_{2b}}_{,HL}$ in $0.5 > R \geqslant 0$. Figure 5c shows the over-all picture for the high-low loading process including the transition period. The fitted reliability curves agree with simulation data except for the tail of the low stress section, say

$0.2 > R$. Obviously, there is an increase of mean fatigue cycles in the transition period, i.e. $\overline{n_{2a}}_{,HL}$, but a decrease in the low stress section. The fitted curve of reliability is little higher than data in the tail part, thus it needs an additive modification in the parameter c_2 for better fitting.

Figure 6 presents the degradation of the mean residual strength of survivals in the high-low loading process, by Eq. (A.4), over the reliability. It can be seen that the mean residual strength in the high-stress section $(R > 0.5)$ complies with that in the single-stage process with $S = 66.6\,\mathrm{MPa}$. The zoom-out view around the adjustment shows that the mean residual strength is smaller in the low stress section than that in the single-stage process at low-level stress $S = 56.6\,\mathrm{MPa}$. Thus, a modification of cumulative nature is needed as the loading process is adjusted from high-level to low-level stress.

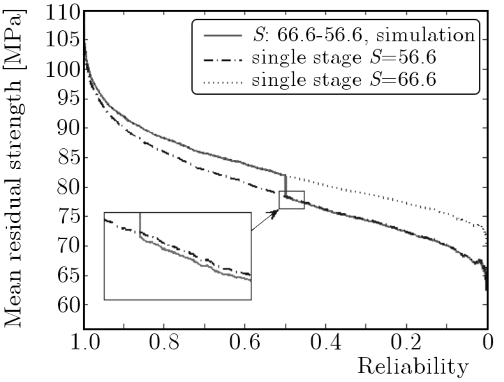

Fig. 6. Variation in the mean residual strength of survivals over the reliability for composites under constant-amplitude cyclic stresses and the high-low loading process shown in Fig. 4

Figure 7 depicts the even better piecewise fitting for the same case as in Fig. 5. It results from the increasing modification of c_2 in Eq. (2.10) with $\zeta = 0.167$ and $\gamma = 2$, which are obtained by fitting the simulated fatigue failure data for every R_a, 10% apart, in $0.9 \geqslant R_a \geqslant 0.1$. The increase of mean fatigue cycles in the transition period appears larger than the decrease in the low stress section. The damage sum D_{HL} calculated by Eq. (2.18) is 1.031.

Figure 8 shows the typical piecewise fitting of the (e_ocp) hazard rate function as given by Eqs. (2.11) and (2.12), for the low-high simulation data, adjusted from $S_1 = 56.6\,\mathrm{MPa}$ to $S_2 = 66.6\,\mathrm{MPa}$ at $R_a = 0.5$. As shown in this figure, except for the abrupt rise at the instant of low-high adjustment, the piecewise fittings are satisfied. It is obvious that the hazard rate is higher in the section of $S_2 = 66.6\,\mathrm{MPa}$ than that in the section of $S_1 = 56.6\,\mathrm{MPa}$.

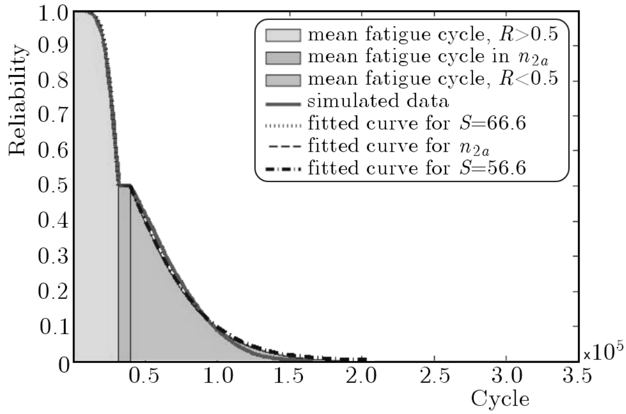

Fig. 7. Piece-wise fitting of the $(e_o cp)$ model for high-low simulated data for G1/Ep$[\pm 45]_S$, from $S_1 = 66.6$ MPa to $S_2 = 56.6$ MPa at $R_a = 0.5$, with $c_2' = 3.5\text{E-}5$

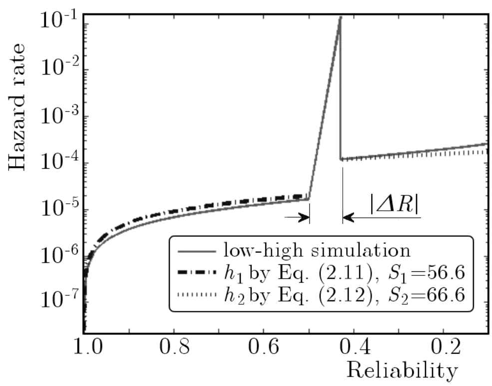

Fig. 8. Typical piecewise curve fitting the $(e_o cp)$ hazard rate function for simulated fatigue data for G1/Ep$[\pm 45]_S$ laminate under low-high loading adjustment, from $S_1 = 56.6$ MPa to $S_2 = 66.6$ MPa at $R_a = 0.5$

Figure 9 displays the piecewise representation of the reliability for the corresponding conditions in Fig. 8. Figure 9a shows the fitting under $S_1 = 56.6$ MPa. The shaded area indicates the mean fatigue cycles $\overline{n_1}_{,LH}$ for $1 \geqslant R > 0.5$. Figure 9b shows the fitting under $S_2 = 66.6$ MPa, where the shaded area indicates the mean fatigue cycles $\overline{n_2}_{,LH}$ for $R < (0.5 - |\Delta R|)$. The area under the fitted curve from $R = 0.5$ to $(0.5 - |\Delta R|)$ denotes the decrease of the mean fatigue cycles at the low-high loading adjustment. As shown in Fig. 9c, the comparison between the piecewise fitted curves and the simulation data is satisfactory. The damage sum D_{LH} calculated by Eq. (2.22) is 0.975.

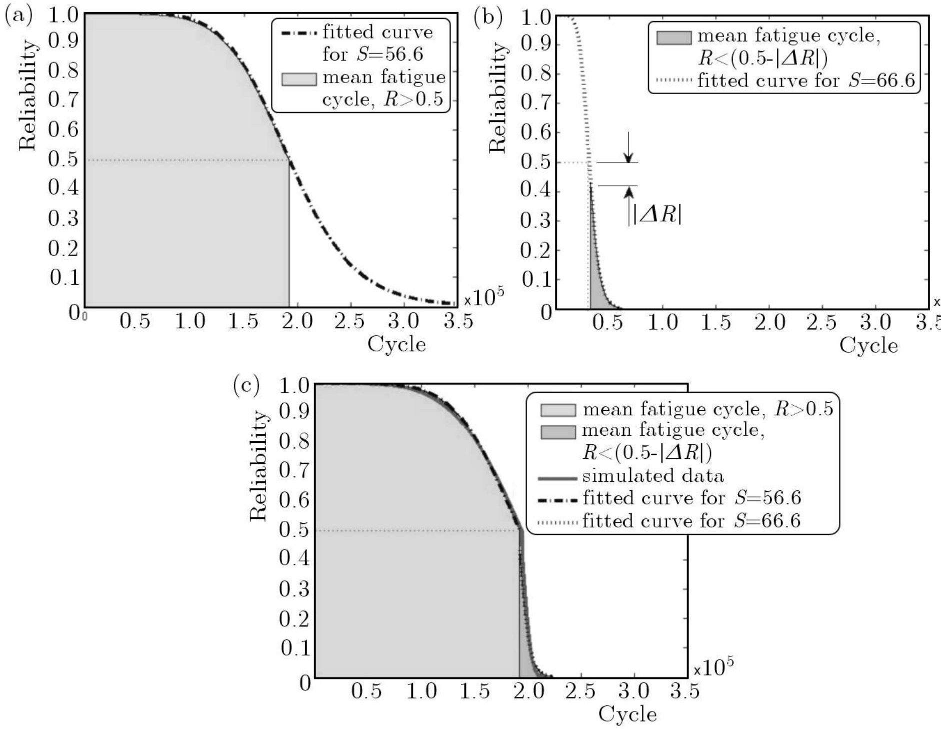

Fig. 9. Typical piecewise fitting of the (e_ocp) model for low-high simulation data for G1/Ep$[\pm 45]_S$ laminate: (a) under $S = 56.6$ MPa; (b) under $S = 66.6$ MPa; (c) adjusted from $S_1 = 6.6$ MPa to $S_2 = 66.6$ MPa at $R_a = 0.5$

Figure 10 depicts the variation in the damage sums when all composite specimens fail under two-stage fatigue loading with various values of R_a. As shown in Fig. 10a, the damage sum D_{HL} obtained from Eq. (2.21) is greater than unity under high-low fatigue loading. This value approaches a peak when R_a is near 0.9. With $S_2 = 56.6$ MPa, the larger S_1 the larger D_{HL}. As commented on Fig. 7, the positive deviation from unity is mainly due to the term $\overline{n_{2a}}_{,HL}/\overline{N_2}$ in Eq. (2.21). Hence, the trend of variation of D_{HL} over R_a complies with that of $\overline{n_{2a}}_{,HL}$, as shown in Fig. 3. It can be seen in Fig. 10b that D_{LH} is smaller than unity for composites experiencing the low-high fatigue loading process. As commented on Fig. 9b, the negative deviation from unity results from the decrease in the mean fatigue cycles from $R = R_a$ to $(R_a - |\Delta R|)$. This deviation decreases to the lowest level when R_a around 0.9. With $S_1 = 56.6$ MPa, a larger S_2 leads to a smaller D_{LH}.

This paper presents an easy method to describe accurately the overall dynamical reliability of composites under two-stage fatigue loading processes by

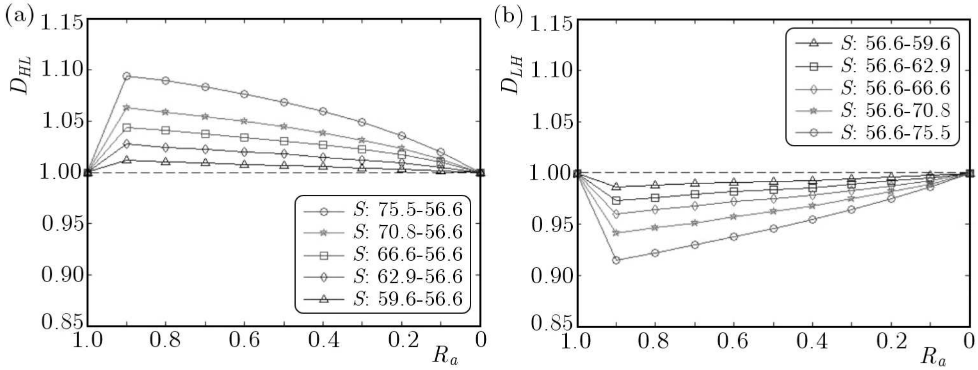

Fig. 10. Variation in the linear damage sum from the (e_ocp) model for simulation data over the reliability at loading adjustment for: (a) high-low cases; (b) low-high cases

a simple method of piecewise combination of the (e_ocp) model. The derivation of the transition period and reliability drop is a pioneer research concerning the effect of fatigue loading adjustment on the dynamical reliability and linear damage sum of composites. The transition period can also be applied in the stress screening of newly developed products of composite materials. The positive and negative deviation of the linear damage sum from unity in high-low and low-high loading, respectively, corresponds with the results of most previous researches of the load sequence effect (Broutman and Sahu, 1972; Yang and Jones, 1980, 1981, 1983; Gamstedt and Sjögren, 2002; Found and Quaresimin, 2003). Furthermore, this paper shows how and how much the stress level and instant of adjustment affect the linear damage sum of composites. The above results can be helpful for the designing and maintenance of the structure of composite materials.

5. Conclusions

Based on the (e_ocp) model for finding the hazard rate, the fitted reliabilities for a single-stage loading process are successfully extended to cases of two-stage loading in combination with the predicted transition period or reliability drop. A better fit can be obtained for the process-dependent decay factor c_2 when c_2' is replaced with a modification for the second stage, especially for a high-low fatigue process. Although the failure does not occur during n_{2a}, the imbedded strength degradation still continues.

As all specimens fail, the linear damage sum is observed to be larger than unity in the high-low loading process, and smaller than unity in the low-high cases. The sums always rise to a peak near $R_a = 0.9$ for high-low cases, and fall to a low value for low-high cases. With a fixed low-level maximum cyclic stress, the deviation of the fatigue damage sum from unity becomes larger as the high-level stress increases.

Appendix

The transition period at the high-low fatigue loading adjustment is expressed as

$$n_{2a} = \frac{(S_1^\omega - S_2^\omega)\Big[S_2^r - \beta^r(-\ln R_a)^{\frac{r}{\alpha}}\Big]}{\beta^r K S_2^b \Big[S_2^\omega - \beta^\omega(-\ln R_a)^{\frac{\omega}{\alpha}}\Big]} \tag{A.1}$$

where α and β are the shape parameter and scale parameter of the Weibull static strength distribution of composites. K and b are the parameters in the S-N curve equation, $KS^bN^* = 1$, where N^* is the characteristic fatigue life associated with S. $r = \alpha/\alpha_f$ is the ratio of α to the shape parameter α_f of the distribution function of the fatigue life N (Yang and Jones,1980, 1981, 1983)

$$P[N \leqslant n] = \begin{cases} 1 - \exp\Big\{-\Big[\dfrac{n}{N^*} + \Big(\dfrac{S}{\beta}\Big)^r\Big]^{\alpha_f}\Big\} & \text{for} \quad n \geqslant 0 \\ 0 & \text{for} \quad n < 0 \end{cases} \tag{A.2}$$

ω is the degradation rate parameter in the residual strength equation

$$X_S^\omega(n) = X^\omega(0) - \frac{X^\omega(0) - S^\omega}{X^r(0) - S^r}\beta^r K S^b n \tag{A.3}$$

where $X(0)$ is the random static strength, and $X_S(n)$ is the random residual strength after n cycles under S. For a two-stage fatigue loading process, the equation of residual strength is

$$X_{S_1+S_2}^\omega(n_1+n_2) = X^\omega(0) - \frac{X^\omega(0) - S_1^\omega}{X^r(0) - S_1^r}\beta^r K S_1^b n_1 - \frac{X^\omega(0) - S_2^\omega}{X^r(0) - S_2^r}\beta^r K S_2^b n_2 \tag{A.4}$$

where $X_{S_1+S_2}(n_1+n_2)$ is the random residual strength after n_1 cycles under S_1 plus n_2 cycles under S_2.

The reliability drop at the low-high fatigue loading adjustment is

$$|\varDelta R| = \exp\Big[-\Big(\frac{x_1}{\beta}\Big)^{\alpha}\Big] - \exp\Big[-\Big(\frac{x_2}{\beta}\Big)^{\alpha}\Big] \tag{A.5}$$

where x_1 is the static strength of the specimens the residual strength of which degrades to S_1 at $n_{1,LH}$ cycles under S_1; and x_2 the static strength of the specimens the residual strength of which degrades to S_2 at $n_{1,LH}$ cycles under S_1. The static strength x_1 is in the form

$$x_1 = (n_{1,LH}\beta^r K S_1^b + S_1^r)^{\frac{1}{r}} \tag{A.6}$$

and x_2 can be obtained by solving the following equation numerically

$$x_2^{r+\omega} - S_2^{\omega} x_2^r - (S_1^r + K\beta^r S_1^b n_{1,LH})x_2^{\omega} + S_2^{\omega} S_1^r + K\beta^r S_1^{\omega+b} n_{1,LH} = 0 \tag{A.7}$$

References

1. Broutman L.J., Sahu S.A., 1972, A new theory to predict cumulative fatigue damage, [In:] *Fiberglass Reinforced Plastics, Composite Materials: Testing and Design (Second Conference)*, ASTM STP 497, 170-188
2. Chen C.L., Tasi Y.T., Wang K.S., 2009, Characteristics of reliability-dependent hazard rate for composites under fatigue loading, *Journal of Mechanics*, **25**, 2, 117-126
3. Chen C.L., Wang K.S., 2011, Effects of loading adjustment on the reliability degradation of composites, *Science and Engineering of Composite Materials*, **18**, 1/2, 61-67
4. Found M.S., Quaresimin M., 2003, Two-stage loading of woven carbon fiber reinforced laminates, *Fatigue and Fracture of Engineering Materials and Structures*, **26**, 17-26
5. Gamstedt E.K., Sjögren B.A., 2002, An experimental investigation of the sequence effect in block amplitude loading of cross-ply composite laminates, *International Journal of Fatigue*, **24**, 2/4, 437-446
6. Han K., Hamdi M., 1983, Fatigue life scattering of RP/C, *38th Annual RP/CI*
7. Hwang W., Han K.S., 1986, Cumulative damage models and multi-stress fatigue life prediction, *Journal of Composite Materials*, **20**, 125-153
8. Ni K., Zhang S., 2000, Fatigue reliability analysis under two-stage loading, *Reliability Engineering and System Safety*, **68**, 153-158

9. PHILIPPIDIS T.P., PASSIPOULARDIDIS V.A., 2007, Residual strength after fatigue in composites: theory vs. experiment, *International Journal of Fatigue*, **29**, 12, 2104-2116

10. SHIH Y.C., 2000, Study of the relationship between the cumulative failure and system reliability, Master thesis, National Central University, Taiwan, R.O.C.

11. TANAKA S., ICHIKAWA M., AKITA S., 1984, A probabilistic investigation of fatigue life and cumulative cycle ratio, *Engineering Fracture Mechanics*, **20**, 3, 501-513

12. WANG K.S, SHEN Y.C., HUANG J.J., 1997, Loading adjustment for fatigue problem based on reliability consideration, *International Journal of Fatigue*, **19**, 10, 693-702

13. WANG K.S., HSU F.S., LIU P.P., 2002, Modeling the bathtub shape hazard rate function in terms of reliability, *Reliability Engineering and System Safety*, **75**, 397-406

14. WANG K.S., 2011, Study of hazard rate function on the cumulative damage phenomenon, *Journal of Mechanics*, **27**, 1, 47-55

15. YANG J.N., JONES D.L., 1980, Effect of load sequence on the statistical fatigue of composites, *AIAA (American Institute of Aeronautics and Astronautics) Journal*, **18**, 12, 1525-1531

16. YANG J.N., JONES D.L., 1981, Load sequence effects on the fatigue of unnotched composite laminates, [In:] *Fatigue of Fibrous Composite Materials*, Lauraitis K.N. (Edit.), ASTM STP 723, Philadelphia, 213-232

17. YANG J.N., JONES D.L., 1983, Load sequence effects on graphite/epoxy [G35]2s, [In:] *Long Term Behavior of Composites*, O'Brien T.K. (Edit.), ASTM STP 813, Philadelphia, 246-262

Konstrukcja funkcji ryzyka uszkodzeń kawałkami zależnej od niezawodności dla kompozytów poddanych różnym scenariuszom obciążenia zmęczeniowego

Streszczenie

W oparciu o wyznaczony okres przejściowy i spadek niezawodności, artykuł prezentuje metodę określania funkcji ryzyka uszkodzenia kawałkami zależnej od poziomu niezawodności, zwanej (e_ocp) i służącej do modelowania dynamicznej niezawodności dla dwustanowych procesów obciążania zmęczeniowego. Na poczatku, parametry e_o, c, i p dopasowano do danych otrzymanych w drodze symulacji uszkodzeń pod wpływem działania cyklicznych naprężeń o kilku stałych amplitudach. Niezawodność dla obciążeń przechodzących od dużej amplitudy do małej opisano kawałkami zależnymi od poziomu przykładanych naprężeń i odpowiadającymi im wartościami e_o, c, i p. Wynosi ona R_a w okresie przejściowym, gdzie R_a jest niezawodnością, przy której poziom naprężeń jest zmieniany. Niezawodność przy obciążeniu rosnącym wyznaczono, odejmując część jej spadku przy R_a od kawałkami dopasowanych krzywych. Zaproponowany sposób opisu niezawodności sukcesywnie weryfikowano. Zaobserwowano, że liniowa suma uszkodzeń przekracza jedność dla scenariusza obciążeń stopniowo malejących i nie osiąga tej wartości w przypadku przeciwnym. Większe różnice w poziomach obciążeń skutkowały w większych odstępstwach liniowej sumy uszkodzeń od jedności. Szczególnie duże zauważono dla $R_a = 0.9$

Manuscript received January 20, 2011; accepted for print April 3, 2011