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SOME EXACT SOLUTIONS FOR OLDROYD-B FLUID DUE TO TIME DEPENDENT PRESCRIBED SHEAR STRESS

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The velocity field and the shear stress corresponding to motion of an Oldroyd-B fluid between two infinite coaxial circular cylinders are established by means of the Hankel transforms. The flow of the fluid is produced due to the time dependent axial shear stress applied on the boundary of the inner cylinder. The exact solutions, presented under a series form, can easily be specialized to give similar solutions for the Maxwell, second grade and Newtonian fluids performing the same motion. Finally, some characteristics of the motion as well as the influence of the material constants on the behavior of the fluid are underlined by graphical illustrations.

Key words: Oldroyd-B fluid, velocity field, time-dependent shear stress, Hankel transform

1. Introduction

During the past two decades, viscoelastic fluids are considered to play a more important and appropriate role in technological applications in comparison with Newtonian fluids. Large industrial materials fall into this category, such as solutions and melts of polymers, soap and cellulose solutions, biological fluids, various colloids and paints, certain oils and asphalts. Thus, due to diversity of fluids in nature, several models have been suggested in the literature. Amongst these, rate type fluids have gained much popularity (Bird and Armstrong, 1987; Dun and Fosdick, 1974; Dun and Rajagopal, 1995; Oldroyd, 1950; Rajagopal and Kaloni, 1989; Rajagopal and Srinivasa, 2000). One of the most popular model for the rate type fluids is known as the the Oldroyd-B fluid model. The flows of Oldroyd-B fluid have been studied in much details, more than most other non-Newtonian fluid models and in complicated flow geometries. The Oldroyd-B fluid can be found frequently in the field of blowing and extrusion molding. Unfortunately, the response of polymeric liquids is so complex that no model can adequately describe their response. As the Oldroyd-B fluid can describe stress-relaxation, creep and the normal stress differences but it cannot describe either shear thinning or shear thickening, a phenomenon that is exhibited by many polymer materials. However, this model can be viewed as one of the most successful model for describing the response of some polymeric liquids.

The exact analytical solution for the flow of an Oldroyd-B fluid was given by Waters and King (1970), Rajagopal and Bhatnagar (1995), Fetecau (2003, 2004), Fetecau and Fetecau (2003, 2005). Other interesting results were obtained by Georgiou (1996) for small one-dimensional perturbations and for the limiting case of zero Reynolds number, unsteady unidirectional transient flows of the Oldroyd-B fluid in an annular pipe, and unsteady transient rotational flows of the Oldroyd-B fluid in an annular pipe are given by Tong and Liu (2005) and Tong and Wang (2005). Some simple flows and exact solutions of the Oldroyd-B fluid are also examined by Hayat *et al.* (2001, 2004). Wood (2001) has considered the general case of helical flow of the Oldroyd-B fluid, due to combined action of rotating cylinders (with constant angular velocities) and a constant axial pressure gradient. Following Rahaman and Ramkissoon (1995), he assumes that the velocity profiles have Taylor series expansions and uses this assumption to derive a second initial condition. Fetecau *et al.* (2007) obtained the velocity fields and the associated tangential stresses corresponding to some helical flows of Oldroyd-B fluids in a series form in terms of the Bessel functions. However, it is important to point out that all the above mentioned works dealt with problems in which the velocity is given on the boundary. To the best of our knowledge, the first exact solutions for flows into cylindrical domains, when the shear stress is given on the boundary, are those obtained by Waters and King (1970) for Oldroyd-B fluids. Similar solutions, corresponding to a time-dependent shear stress on the boundary, have been recently obtained in Fetecau *et al.* (2009a,b, 2010), Jamil *et al.* (2011), Nazar *et al.* (2011), Siddique and Sajid (2011).

In this note, exact solutions corresponding to the axial flow of an Oldroyd-B fluid in an annular region between two infinite circular cylinders are established. In order to produce flow of the fluid, the boundary of the inner cylinder is subject to a time-dependent axial shear stress. The solutions that have been here obtained tend to the similar solutions for the Maxwell, second grade and Newtonian fluids by taking appropriate limits. Finally, the influence of physical constants on the velocity profile and the shear stress is shown by graphical illustrations.

2. Basic flow equations

The Cauchy stress $\mathbf{T}$ in an incompressible Oldroyd-B fluid is given by (Fetecau *et al.*, 2009a,b, 2010; Jamil *et al.*, 2011; Nazar *et al.*, 2011; Siddique and Sajid, 2011)

$$\begin{aligned} &\mathbf{T} = -p\mathbf{I} + \mathbf{S} \\ &\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^\top) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^\top)] \end{aligned} \tag{2.1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericsen tensor, μ is the dynamic viscosity of the fluid, λ and λ_r are relaxation and retardation times, the superscript $(\cdot)^\top$ indicates the transpose operation, and the superposed dot indicates the material time derivative. The model characterized by constitutive equations (2.1) contains as special cases the upper-convected Maxwell model for $\lambda_r \to 0$ and the Newtonian fluid model for $\lambda_r \to 0$ and $\lambda \to 0$. In some special flows, as those to be considered here, the governing equations for an Oldroyd-B fluid resemble those for a fluid of the second grade. For the problem under consideration, we shall assume a velocity field and an extra-stress of the form

$$\boldsymbol{V} = \boldsymbol{V}(r,t) = v(r,t)\boldsymbol{e}_z \qquad \mathbf{S} = \mathbf{S}(r,t) \tag{2.2}$$

where $\boldsymbol{e}_z$ is the unit vector in the z-direction of the system of cylindrical coordinates r, θ and z. For such flows, the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\boldsymbol{V}(r,0) = \mathbf{0} \qquad \mathbf{S}(r,0) = \mathbf{0} \tag{2.3}$$

and Eqs. (2.1) and (2.2) imply $S_{rr} = S_{r\theta} = S_{\theta z} = S_{\theta\theta} = 0$.

In the absence of body forces and a pressure gradient in the axial direction, the balance of linear momentum and constitutive equation (2.1) lead to the relevant equations

$$\begin{aligned}\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau(r,t) &= \mu\Big(1+\lambda_r\frac{\partial}{\partial t}\Big)\frac{\partial}{\partial r}v(r,t)\\ \rho\frac{\partial}{\partial t}v(r,t) &= \Big(\frac{\partial}{\partial r}+\frac{1}{r}\Big)\tau(r,t)\end{aligned} \tag{2.4}$$

where ρ is the constant density of the fluid and $\tau = S_{rz}$ is the shear stress that is different from zero.

Eliminating τ between Eqs. (2.4), we obtain the governing equation

$$\lambda\frac{\partial^2}{\partial t^2}v(r,t)+\frac{\partial}{\partial t}v(r,t) = \Big(\nu+\alpha\frac{\partial}{\partial t}\Big)\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Big)v(r,t) \tag{2.5}$$

where $\alpha = \nu\lambda_r$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. Partial differential equation (2.5), with adequate initial and boundary conditions, can be solved in principle by several methods, their effectiveness strictly depending on the domain of definition. In our case, the integral transforms technique presents a systematic, efficient and powerful tool. The Hankel transform can be used to eliminate the spatial variable.

3. Flow through an infinite annular region

Suppose that an incompressible Oldroyd-B fluid at rest is situated in the annular region between two infinite coaxial circular cylinders of radii R_1 and $R_2(> R_1)$. At time $t = 0^+$ the inner cylinder is pulled along its axis with a time-dependent shear stress

$$\tau(R_1,t) = f[t-\lambda(1-\mathrm{e}^{-\frac{t}{\lambda}})] \qquad t>0 \tag{3.1}$$

where f is a constant, while the outer one is held fixed. Due to shear, the fluid between cylinders is gradually moved, its velocity being of form (2.2). The governing equations are given by Eqs. $(2.4)_1$ and (2.5) and the appropriate initial and boundary conditions are

$$v(r,0) = \frac{\partial v(r,0)}{\partial t} = 0 \qquad \tau(r,0) = 0 \qquad r\in[R_1,R_2] \tag{3.2}$$

and for $t > 0$

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau(r,t)\Big|_{r=R_1} = \mu\Big(1+\lambda_r\frac{\partial}{\partial t}\Big)\frac{\partial v(r,t)}{\partial r}\Big|_{r=R_1} = ft \qquad v(R_2,t)=0 \tag{3.3}$$

In order to solve this problem, we shall use the Hankel transforms.

3.1. Calculation of the velocity field

In order to determine the velocity field, let us denote by (Tong and Wang, 2005; Debnath and Bhatta, 2007)

$$v_{nH}(t) = \int\limits_{R_1}^{R_2} rv(r,t)B(r,r_n)\;dr \qquad n=1,2,3,\dots \tag{3.4}$$

the finite Hankel transform of the function $v(r,t)$, where

$$B(r,r_n) = J_0(rr_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_0(rr_n)$$

r_n being the positive roots of the equation $B(R_2,r) = 0$ and $J_p(\cdot)$, $Y_p(\cdot)$ are the Bessel functions of the first and second kind of the order p. Using Eqs. (3.3), and the known relation

$$B(R_1,r_n) = J_0(R_1r_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_0(R_1r_n) = -\frac{2}{\pi R_1 r_n}$$

we can prove that

$$\int\limits_{R_1}^{R_2} r\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\Big)v(r,t)B(r,r_n)\;dr = \frac{2}{\pi r_n}\frac{\partial v(R_1,t)}{\partial r} - r_n^2 v_{nH}(r_n,t) \tag{3.5}$$

Furthermore, the inverse Hankel transform is (Tong and Wang, 2005; Bandelli and Rajagopal, 1995)

$$v(r,t) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_0^2(R_2r_n)B(r,r_n)}{J_1^2(R_1r_n)-J_0^2(R_2r_n)}\,v_{nH}(t) \tag{3.6}$$

and

$$\int\limits_{R_1}^{R_2} r\log\Big(\frac{r}{R_2}\Big)B(r,r_n)\;dr = \frac{2}{\pi R_1 r_n^3} \tag{3.7}$$

Multiplying Eq. (2.5) by $rB(r,r_n)$, integrating the result with respect to r from R_1 to R_2, and using boundary condition $(3.3)_1$ and identity (3.5), we find that

$$\lambda\ddot{v}_{nH}(t)+(1+\alpha r_n^2)\dot{v}_{nH}(t)+\nu r_n^2 v_{nH}(t)=\frac{2ft}{\pi\rho r_n}\qquad t>0 \tag{3.8}$$

From (3.2) it also results that

$$v_{nH}(0)=\dot{v}_{nH}(0)=0 \tag{3.9}$$

The solution to linear ordinary differential equation (3.8), with initial conditions (3.9), is given by

$$v_{nH}(t)=\frac{2f}{\pi\mu r_n^3}\Big[t-\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}-\frac{1+\alpha r_n^2}{\nu r_n^2}\Big(1-\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}\Big)\Big] \tag{3.10}$$

where

$$q_{1n},q_{2n}=\frac{-(1+\alpha r_n^2)\pm\sqrt{(1+\alpha r_n^2)^2-4\nu\lambda r_n^2}}{2\lambda}$$

Finally, applying the inverse Hankel transform formula and using Eq. (3.7), we find for the velocity field $v(r,t)$ the simple expression

$$\begin{aligned}v(r,t)=&\frac{f}{\mu}(t-\lambda_r)R_1\log\Big(\frac{r}{R_2}\Big)-\frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n^3[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\\&+\frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\\&\cdot\Big(\frac{1+\alpha r_n^2}{\nu r_n^2}\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}-\frac{\mathrm{e}^{q_{2n}t}-\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}\Big)\end{aligned} \tag{3.11}$$

or equivalently

$$\begin{aligned}v(r,t)=&\frac{f}{\mu}(t-\lambda_r)R_1\log\Big(\frac{r}{R_2}\Big)\\&-\frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n^3[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\Big(1-\lambda\frac{q_{1n}^2\mathrm{e}^{q_{2n}t}-q_{2n}^2\mathrm{e}^{q_{1n}t}}{q_{2n}-q_{1n}}\Big)\end{aligned} \tag{3.12}$$

3.2. Calculation of the shear stress

Solving Eq. $(2.4)_1$ with respect to $\tau(y,t)$ and taking into account Eq. $(3.2)_3$, we find that

$$\tau(r,t)=\frac{\mu}{\lambda}\mathrm{e}^{-\frac{t}{\lambda}}\int_0^t\mathrm{e}^{\frac{s}{\lambda}}\Big(1+\lambda_r\frac{\partial}{\partial s}\Big)\partial_r v(r,s)\,ds \tag{3.13}$$

Substituting (3.12) into (3.13), we obtain, after lengthy but straightforward computations, a suitable form for the shear stress

$$\begin{aligned}\tau(r,t) &= f\Big(\frac{R_1}{r}\Big)[t-\lambda(1-\mathrm{e}^{-\frac{t}{\lambda}})] + \frac{\pi f}{\nu}(1-\mathrm{e}^{-\frac{t}{\lambda}})\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\\ &+\frac{\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\frac{\lambda}{q_{2n}-q_{1n}}\Big[\frac{q_{2n}^2(\mathrm{e}^{q_{1n}t}-\mathrm{e}^{-\frac{t}{\lambda}})}{1+\lambda q_{1n}}\\ &-\frac{q_{1n}^2(\mathrm{e}^{q_{2n}t}-\mathrm{e}^{-\frac{t}{\lambda}})}{1+\lambda q_{2n}}+\lambda_r q_{1n}q_{2n}\Big(\frac{q_{2n}(\mathrm{e}^{q_{1n}t}-\mathrm{e}^{\frac{-t}{\lambda}})}{1+\lambda q_{1n}}-\frac{q_{1n}(\mathrm{e}^{q_{2n}t}-\mathrm{e}^{-\frac{t}{\lambda}})}{1+\lambda q_{2n}}\Big)\Big]\end{aligned} \tag{3.14}$$

where

$$\widetilde{B}(r,r_n) = J_1(rr_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_1(rr_n) \tag{3.15}$$

Of course, Eq. (3.14) can further be processed to give the simple form

$$\begin{aligned}\tau(r,t) &= f\Big(\frac{R_1}{r}\Big)[t-\lambda(1-\mathrm{e}^{-\frac{t}{\lambda}})]\\ &+\frac{\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\Big(1-\frac{q_{2n}\mathrm{e}^{q_{1n}t}-q_{1n}\mathrm{e}^{q_{2n}t}}{q_{2n}-q_{1n}}\Big)\end{aligned} \tag{3.16}$$

4. Limiting cases

4.1. Maxwell fluid

Taking the limit of Eqs. (3.12) and (3.16) as $\lambda_r \to 0$, we attain to the solutions

$$\begin{aligned}v_M(r,t) &= \frac{ft}{\mu}R_1\log\Big(\frac{r}{R_2}\Big)\\ &-\frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n^3[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\Big(1-\lambda\frac{q_{5n}^2\mathrm{e}^{q_{6n}t}-q_{6n}^2\mathrm{e}^{q_{5n}t}}{q_{6n}-q_{5n}}\Big)\\ \tau_M(r,t) &= f\Big(\frac{R_1}{r}\Big)[t-\lambda(1-\mathrm{e}^{-\frac{t}{\lambda}})]\\ &+\frac{\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1r_n)-J_0^2(R_2r_n)]}\Big(1-\frac{q_{6n}\mathrm{e}^{q_{5n}t}-q_{5n}\mathrm{e}^{q_{6n}t}}{q_{6n}-q_{5n}}\Big)\end{aligned} \tag{4.1}$$

corresponding to a Maxwell fluid performing the same motion. Into the above relations

$$q_{5n}, q_{6n} = \frac{-1\pm\sqrt{1-4\nu\lambda r_n^2}}{2\lambda}$$

4.2. Second grade fluid

By now letting $\lambda \to 0$ into Eqs. (3.12) and (3.16), the similar solutions

$$\begin{aligned} v_{SG}(r,t) &= \frac{f}{\mu}(t-\lambda_r)R_1 \log\Big(\frac{r}{R_2}\Big) \\ &- \frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r,r_n)}{r_n^3[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}\Big[1-(1+\alpha r_n^2)\exp\Big(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\Big)\Big] \\ \tau_{SG}(r,t) &= ft\Big(\frac{R_1}{r}\Big) \\ &+ \frac{\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}\Big[1-\exp\Big(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\Big)\Big] \end{aligned} \tag{4.2}$$

corresponding to a second grade fluid are recovered.

4.3. Newtonian fluid

Finally, making λ and $\lambda_r \to 0$ into Eqs. (3.12) and (3.16) or $\lambda \to 0$ into (4.1) respectively, $\lambda_r \to 0$ into (4.2), the solutions for a Newtonian fluid

$$\begin{aligned} v_N(r,t) &= \frac{ft}{\mu}R_1 \log\Big(\frac{r}{R_2}\Big) - \frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r,r_n)}{r_n^3[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}(1-\mathrm{e}^{-\nu r_n^2 t}) \\ \tau_N(r,t) &= ft\Big(\frac{R_1}{r}\Big) + \frac{\pi f}{\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)\widetilde{B}(r,r_n)}{r_n^2[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}(1-\mathrm{e}^{-\nu r_n^2 t}) \end{aligned} \tag{4.3}$$

are obtained. Of course, for the last two cases (Newtonian and second grade fluids), the boundary condition obtained from (3.1) for $\lambda \to 0$ is

$$\tau(R_1,t) = ft \tag{4.4}$$

5. Connection with some similar results from the literature

The unsteady motion through an infinite annular region due to the inner cylinder that applies a constant longitudinal shear f to a second grade fluid has been studied by Bandelli and Rajagopal (1995). The velocity field that was obtained is

$$v_{SG}^0(r,t) = \frac{f}{\mu}R_1 \log\Big(\frac{r}{R_2}\Big) - \frac{\pi f}{\mu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2 r_n)B(r,r_n)}{r_n[J_1^2(R_1 r_n)-J_0^2(R_2 r_n)]}\exp\Big(-\frac{\nu r_n^2 t}{1+\beta r_n^2}\Big) \tag{5.1}$$

where $\beta = \alpha_1/\mu$, α_1 is a material constant and r_n are the same roots as before. Simple analysis shows that the two solutions $v^0_{SG}(r,t)$ and $v_{SG}(r,t)$, corresponding to the shear stress f, respectively ft on the boundary, are related by the integral relation

$$v_{SG}(r,t) = \int_0^t v^0_{SG}(r,s)\, ds \tag{5.2}$$

if $\alpha = \beta = \nu\lambda_r$. Of course, simple computations also show that

$$\tau_{SG}(r,t) = \int_0^t \tau^0_{SG}(r,s)\, ds \tag{5.3}$$

Consequently, the solutions $v_{SG}(r,t)$ and $\tau_{SG}(r,t)$, as well as $v_N(r,t)$ and $\tau_N(r,t)$, corresponding to the motion of a second grade or Newtonian fluid due to the inner cylinder that applies a shear stress ft to the fluid, can immediately be obtained by simple integration if the similar solutions $v^0_{SG}(r,t)$, $\tau^0_{SG}(r,t)$, $v^0_N(r,t)$ and $\tau^0_N(r,t)$ corresponding to a constant shear on the boundary, are known.

6. Numerical results and conclusions

The purpose of this note is to provide exact solutions for the velocity field and the shear stress corresponding to the axial flow of an Oldroyd-B fluid between two infinite circular cylinders, the inner cylinder being subject to a time-dependent shear stress and the outer one at rest. These solutions, obtained by means of the finite Hankel transforms, are presented under a series form in terms of the Bessel functions $J_0(\cdot)$, $J_1(\cdot)$, $Y_0(\cdot)$ and $Y_1(\cdot)$. Direct computations show that they satisfy both the governing equations and all initial and boundary conditions. Furthermore, for $\lambda_r \to 0$ or $\lambda \to 0$, general solutions (3.12) and (3.16) reduce to the corresponding solutions for Maxwell fluids and second grade fluids, respectively. The last solutions, as well as the general solutions, can easily be specialized to give the similar solutions for Newtonian fluids performing the same motion.

All solutions are presented as a sum between large-time and transient solutions. Simple analysis shows that

$$v_{LT}(r,t) = \frac{f}{\mu}(t-\lambda_r)R_1\log\Big(\frac{r}{R_2}\Big) - \frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n^3[J_1^2(R_1r_n)-J_0^2(R_2r_n)]} = v_{SGLT}(r,t) \tag{6.1}$$

$$v_{MLT}(r,t) = \frac{ft}{\mu}R_1\log\Big(\frac{r}{R_2}\Big) - \frac{\pi f}{\mu\nu}\sum_{n=1}^{\infty}\frac{J_0^2(R_2r_n)B(r,r_n)}{r_n^3[J_1^2(R_1r_n)-J_0^2(R_2r_n)]} = v_{NLT}(r,t)$$

Consequently, as expected, for large times the velocity of an Oldroyd-B fluid can be well enough approximated by that of a second grade fluid. The last relation shows that for Maxwell fluids the non-Newtonian effects on the fluid motion disappear in time. As regards the shear stresses, it is clearly that $\tau_{LT}(r,t) = \tau_{MLT}(r,t)$ and $\tau_{SGLT}(r,t) = \tau_{NLT}(r,t)$. This is not a surprise, because the shear stress on the inner cylinder is the same for Oldroyd-B and Maxwell fluids, respectively for the second grade and Newtonian fluids.

Finally, in order to reveal some relevant physical aspects of the obtained results, diagrams of the velocity field and the shear stress have been drawn against r for different values of t and material costants. Figures 1a and 1b clearly show that the velocity $v(r,t)$, as well as the shear stress $\tau(r,t)$ in absolute value, is an increasing function of t. The influence of the relaxation and retardation times λ and λ_r on the fluid motion is shown into Fig. 2 and 3. The velocity of the fluid, as well as the shear stress in absolute value, is a decreasing function of λ and λ_r. The effect of the kinematic viscosity, as it results from Figs. 4a and 4b, is opposite to that of λ and λ_r. The velocity of the fluid, as expected, increases for increasing ν.

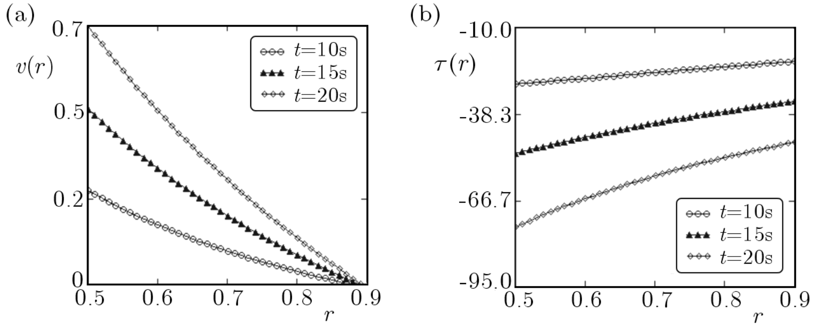

Fig. 1. Profiles of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (3.12) and (3.16), for $R_1 = 0.5$, $R_2 = 0.9$, $f = -5$, $\nu = 0.0357541$, $\mu = 32$, $\lambda = 5$, $\lambda_r = 3$ and different values of t

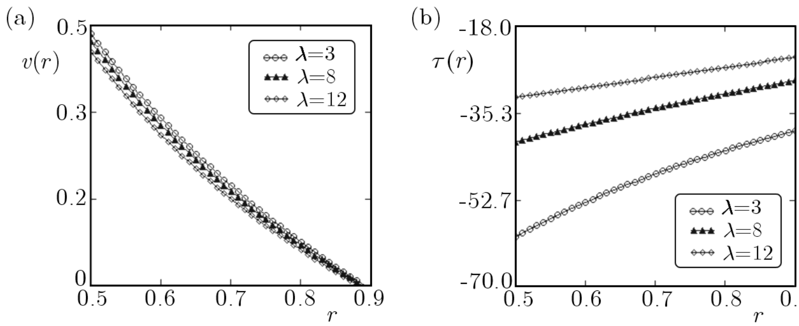

Fig. 2. Profiles of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (3.12) and (3.16), for $R_1 = 0.5$, $R_2 = 0.9$, $f = -5$, $\nu = 0.0357541$, $\mu = 32$, $\lambda_r = 3$, $t = 15\,\mathrm{s}$, and different values of λ

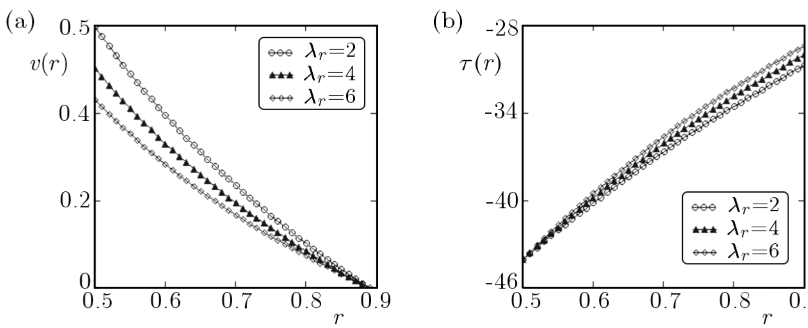

Fig. 3. Profiles of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (3.12) and (3.16), for $R_1 = 0.5$, $R_2 = 0.9$, $f = -5$, $\nu = 0.0357541$, $\mu = 32$, $\lambda = 7$, $t = 15\,\mathrm{s}$, and different values of λ_r

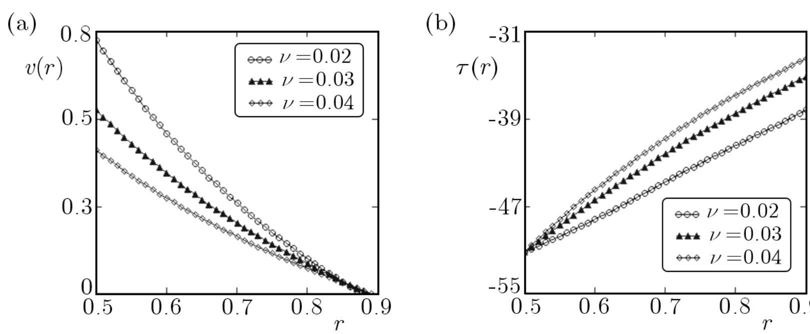

Fig. 4. Profiles of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$ given by Eqs. (3.12) and (3.16), for for $R_1 = 0.5$, $R_2 = 0.9$, $f = -5$, $\rho = 895$, $\lambda = 5$, $\lambda_r = 3$, $t = 15\,\mathrm{s}$, and different values of ν

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Wybrane rozwiązania dokładne dla cieczy Oldroyda-B przy zadanej funkcji naprężeń stycznych zależnej od czasu

Streszczenie

Pole prędkości i pole rozkładu naprężeń stycznych wywołanych ruchem cieczy Oldroyda-B umieszczonej między dwoma koncentrycznymi cylindrami wyznaczono za pomocą transformaty Hankela. Przepływ cieczy wywołano zależnym od czasu naprężeniem stycznym od zewnętrznej ściany cylindra wewnętrznego. Uzyskane rozwiązanie dokładne, ujęte w formie rozwinięcia w szereg, może łatwo być zastosowane dla przypadków szczególnych cieczy Maxwella, cieczy drugiego stopnia i nienewtonowskich przy tych samych warunkach przepływu. Na zakończenie rozważań, przedstawiono graficznie charakterystyki ruchu cieczy i wpływ parametrów materiałowych na jej zachowanie.

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