JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
51, 2, pp. 287-296, Warsaw 2013

APPLICATION OF HIGHER ORDER HAMILTONIAN APPROACH TO NONLINEAR VIBRATING SYSTEMS

Hassan Askari, Zia Saadat Nia

Iran University of Science and Technology, Center of Excellence in Railway Transportation, School of Railway Engineering, Tehran, Iran

Ahmet Yildirim

Ege University, Department of Mathematics, Bornova, Turkey; e-mail: ahmet.yildirim@ege.edu.tr

Mohammed K. Yazdi

Iran University of Science and Technology, School of Mechanical Engineering, Tehran, Iran

Yasir Khan

Zheijiang University, Department of Mathematics, Hangzhou, China

The higher order Hamiltonian approach is utilized to elicit approximate solutions for two nonlinear oscillation systems. Frequency-amplitude relationships and the model of buckling of a column and mass-spring system are scrutinized in this paper. First, second and third approximate solutions of examples are achieved, and the frequency responses of the systems are verified by exact numerical solutions. According to the numerical results, we can conclude that the Hamiltonian approach is an applicable method for solving the nonlinear equations, and the accuracy of this method in the second and third approximates is very high and reliable. The achieved results of this paper demonstrate that this method is powerful and uncomplicated for solving of sophisticated nonlinear problems.

Key words: higher order Hamiltonian approach, Duffing equation, analytical solutions

1. Introduction

Since the nonlinear science has been emerged in real world uses, there is a cause for increasing attention of scientists and engineers in analytical approaches for nonlinear problems (He, 2006). Recently, many scientists have proposed and modified a lot of methods for solving nonlinear equations (Nayfeh and Mook, 1979). He (2002) have invented several non-perturbative approaches such as energy balance method (EBM), variational approach (He, 2007), max-min approach (He, 2008b), Hamiltonian approach (He, 2010) and frequency amplitude formulation (He, 2008a). Based on He's methods, many researchers have evaluated diverse kinds of nonlinear problems. For instance, D.D. Ganji *et al.* (2010) and S.S. Ganji *et al.* (2009) used energy balance method for solving Van der Pol damped equations and relativistic oscillator. Momeni *et al.* (2011) and Ozis and Yildirim (2007) employed EBM for solving the Duffing harmonic equation. Simiraly, a nonlinear oscillator with discontinuity was analyzed by D.D. Ganji *et al.* (2009) by means of this approach. Also, Younesian *et al.* (2010a) analyzed the generalized Duffing equation by it. The variational approach was applied for solving the relativistic oscillator (He, 2007), generalized Duffing equation (Younesian *et al.*, 2010a), oscillator with a fractional power (Younesian *et al.*, 2010b), Duffing harmonic oscillator (Askari *et al.*, 2010). The frequency amplitude formulation was incorporated by Cai and Wu (2009), Younesian *et al.* (2010a), Kalami *et al.* (2010), Ren *et al.* (2009), Zhang *et al.* (2009) and Zhao (2009) for solving the relativistic harmonic oscillator, generalized Duffing equation, autonomous conservative nonlinear oscillator, nonlinear oscillator with discontinuity, Schrödinger equation and nonlinear oscillator with an irrational force, respectively. Moreover, the max-min approach was used for analyzing the relativistic oscillator (Shen and Mo, 2009), buckling of a column (Ganji *et al.*, 2011), two mass spring system (Ganji *et al.*, 2011), nonlinear oscillator with discontinuity (Zeng, 2009) and a nonlinear oscillation system of motion of a rigid rod rocking back[(Ganji *et al.*, 2010). In this paper, two kinds of systems with the same form of nonlinear equation are analyzed. Figure 1 describes a model of buckling of a column (Nayfeh and Mook, 1979). The vibration of this system was investigated by Ganji *et al.* (2011) and Nayfeh and Mook (1979).

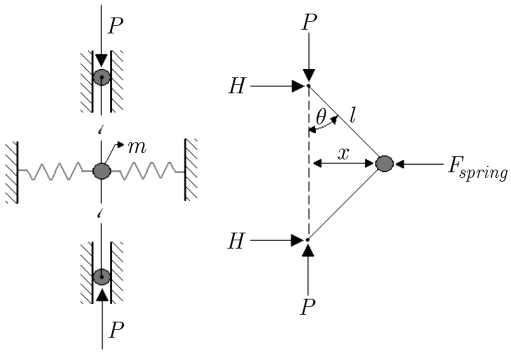

Fig. 1. Model of buckling of a column (Nayfeh and Mook, 1979)

Figure 2 shows the physical model of Duffing equation with a constant coefficient. This system was examined by Mehdipour *et al.* (2010) by means of of the energy balance method.

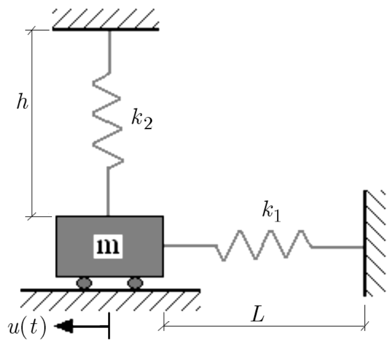

Fig. 2. Physical model of Duffing equation (Rao, 2006)

In the present work, the Hamiltonian approach is used for solving the governing equations of the above problems. This method was invented by J.H. He, and it has been used for evaluating a large number of nonlinear problems. The nonlinear oscillator with a fractional power (Cveticanin, 2010), nonlinear oscillator with discontinuity (Yildrim *et al.*, 2011c), nonlinear oscillator with rational and irrational elastic forces (Yildrim *et al.*, 2011a), nonlinear oscillations of a punctual charge in the electric field of a charged ring (Yildrim *et al.*, 2011b), nonlinear vibration of a rigid rod rocking back Khan *et al.*, 2010) have been solved by means of this potent and straightforward method. Furthermore, Yilidrim *et al.* (2012) have demonstrated the relationship of this method with the variational approach. The frequency-amplitude relationship is then obtained in an analytical form. Also, the obtained frequency responses of the systems are compared with the exact numerical solutions. In addition, the achieved results are compared with the results of the max-min approach that were obtained by Ganji *et al.* (2011). Moreover, according to Yildirim *et al.* (2012), it is stated that the variational approach leads to the same results for this systems even for higher order approximations. Furthermore, results of several papers are developed to obtain the frequency amplitude relationship of this system.

2. Mathematical modeling

In this section, we consider a column as shown in Fig. 1. The mass m moves in the horizontal direction only. Using this model that represents the column, we demonstrate how one can study its static stability by determining the nature of the singular point at $x = 0$ of the dynamic equations (Nayfeh and Mook, 1979; Ganji *et al.*, 2011). Avoiding the weight of springs and columns, the governing equation for motion of m is (Nayfeh and Mook, 1979)

$$m\ddot{u} + \Big(k_1 - \frac{2P}{l}\Big)u + \Big(k_3 - \frac{P}{l^3}\Big)u^3 + \ldots = 0 \tag{2.1}$$

where the spring force is given by

$$F_{Spring} = k_1 u + k_3 u^3 + \ldots \tag{2.2}$$

This equation can be put in the general form

$$\ddot{u} + \alpha_1 u + \alpha_3 u^3 + \ldots = 0 \tag{2.3}$$

Also, the g overning equation for the model shown in Fig. 2 is obtained as

$$\ddot{u} + \frac{K_1}{m}u + \frac{K_2}{2mh^2}u^3 = 0 \tag{2.4}$$

3. Solution procedure

Consider the following equation which describes the well known Duffing equation

$$\ddot{u} + \alpha_1 u + \alpha_3 u^3 = 0 \qquad u(0) = A \qquad \dot{u}(0) = 0 \tag{3.1}$$

where for the first system

$$\alpha_1 = \frac{1}{m}\Big(k_1 - \frac{2P}{l}\Big) \qquad \alpha_3 = \frac{1}{m}\Big(k_3 - \frac{p}{l^3}\Big) \tag{3.2}$$

and for the second one

$$\alpha_1 = \frac{K_1}{m} \qquad \alpha_2 = \frac{K_2}{2mh^2} \tag{3.3}$$

Based on the first order of the Hamiltonian approach introduced by He (2010), a solution for Eq. (3.1) is assumed as

$$u = A\cos\omega t \tag{3.4}$$

with satisfying the initial conditions. Its Hamiltonian can be easily obtained, which reads

$$H = \frac{1}{2}\dot{u}^2 + \frac{\alpha_1}{2}u^2 + \frac{\alpha_3}{2}u^4 \tag{3.5}$$

Integrating Eq. (3.6) with respect to time from 0 to $T/4$, we have

$$\widetilde{H}(u) = \int\limits_0^{T/4} \Big(\frac{1}{2}\dot{u}^2 + \frac{\alpha_1}{2}u^2 + \frac{\alpha_3}{2}u^4\Big)\,dt \tag{3.6}$$

Substituting Eq. (3.4) into Eq. (3.6), leads to

$$\begin{aligned}\widetilde{H}(u) &= \int\limits_0^{T/4} \Big(\frac{1}{2}A^2\omega^2\sin^2\omega t + \frac{\alpha_1}{2}A^2\cos^2\omega t + \frac{\alpha_3}{4}A^4\cos^4\omega t\Big)\,dt\\ &= \int\limits_0^{\pi/2} \Big(\frac{1}{2}A^2\omega\sin^2\omega t + \frac{\alpha_1}{2\omega}A^2\cos^2\omega t + \frac{\alpha_3}{4\omega}A^4\cos^4\omega t\Big)\,dt\\ &= \frac{\pi}{8}A^2\omega + \frac{\alpha_1\pi}{8\omega}A^2 + \frac{\alpha_3\pi}{64\omega}A^4\end{aligned} \tag{3.7}$$

Setting

$$\frac{\partial}{\partial A}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) = \frac{\pi}{4}A\omega^2 + \frac{\alpha_1\pi}{4}A + \frac{4\alpha_3\pi}{16}A^3 = 0 \tag{3.8}$$

and consequently, the obtained frequency equals to

$$\omega = \sqrt{\alpha_1 + \frac{3}{4}\alpha_3A^2} \tag{3.9}$$

The energy balance method (Mehdipour *et al.*, 2010), varioational approach (Yildirim *et al.*, 2012), harmonic balance method (Yildirim *et al.*, 2012) and the max-min (Ganji *et al.*, 2011) approach the same to result for this problem.

3.1. Second order Hamiltonian approach

In order to improve the accuracy of this approach, the following periodic solution is considered (Yildirim *et al.*, 2011c; Durmaz *et al.*, 2010)

$$u = a\cos\omega t + b\cos 3\omega t \tag{3.10}$$

where the initial condition is

$$A = a + b \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.6), we obtain

$$\begin{aligned}\widetilde{H}(u) &= \int\limits_0^{T/4} \Big[\frac{1}{2}(a\omega\sin\omega t + 3b\omega\sin 3\omega t)^2 + \frac{1}{2}\alpha_1(a\cos\omega t + b\cos 3\omega t)^2\\ &\quad + \frac{1}{4}\alpha_3(a\cos\omega t + b\cos 3\omega t)^4\Big]\,dt = \int\limits_0^{\pi/2} \Big[\frac{1}{2}\omega(a\sin t + 3b\sin 3t)^2\\ &\quad + \frac{1}{2\omega}\alpha_1(a\cos t + b\cos 3t)^2 + \frac{1}{4\omega}\alpha_3(a\cos t + b\cos 3t)^4\Big]\,dt\\ &= \frac{\pi}{8}\omega(a^2 + 9b^2) + \frac{\pi}{8\omega}\alpha_1(a^2 + b^2) + \frac{\pi}{64\omega}\alpha_3(3a^4 + 4a^3b + 12a^2b^2 + 3b^4)\end{aligned} \tag{3.12}$$

Setting

$$\begin{aligned}\frac{\partial}{\partial a}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) &= -\frac{\pi}{4}a\omega^2 + \frac{\pi}{4}\alpha_1 a + \frac{\pi}{64}\alpha_3(12a^3 + 12a^2b + 24ab^2) = 0\\ \frac{\partial}{\partial b}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) &= -\frac{18\pi b}{8}\omega^2 + \frac{\pi}{4}\alpha_1 b + \frac{\pi}{64}\alpha_3(4a^3 + 24a^2b + 24ab^2) = 0\end{aligned} \tag{3.13}$$

After some mathematical simplifications, it is achieved that

$$a = 0.95714A \qquad b = 0.04289A \tag{3.14}$$

and the frequency-amplitude relationship can be written as

$$\omega_{SHA} = \sqrt{\alpha_1 + 0.7205\alpha_3 A^2} \tag{3.15}$$

3.2. Third order Hamiltonian approach

Consider the following periodic equation as the response to Eq. (3.1)

$$u = a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t \tag{3.16}$$

where

$$A = a + b + c \tag{3.17}$$

Substituting Eq. (22) into Eq. (3.6), we obtain

$$\begin{aligned}\widetilde{H}(u) =& \int\limits_0^{T/4} \Big[\frac{1}{2}(a\omega\sin\omega t + 3b\omega\sin 3\omega t + 5b\omega\sin 5\omega t)^2 \\ &+ \frac{1}{2}\alpha_1(a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t)^2 + \frac{1}{4}\alpha_3(a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t)^4\Big]\, dt \\ &= \int\limits_0^{\pi/2} \Big[\frac{1}{2}\omega(a\sin t + 3b\sin 3t + 5c\cos\omega t)^2 + \frac{1}{2\omega}\alpha_1(a\cos t + \cos 3t + 5c\cos t)^2 \\ &+ \frac{1}{4\omega}\alpha_3(a\cos t + b\cos 3t + 5c\cos\omega t)^4\Big]\, dt = \frac{\pi}{8}\omega(a^2 + 9b^2 + 25c^2) + \frac{\pi}{8\omega}\alpha_1(a^2 + b^2 + c^2) \\ &+ \frac{\pi}{64\omega}\alpha_3(3a^4 + 3b^4 + 3c^4 + 12b^2c^2 + 12ab^2c + 12a^2bc + 4a^3b + 12a^2c^2 + 12a^2b^2)\end{aligned} \tag{3.18}$$

Setting

$$\begin{aligned}\frac{\partial}{\partial a}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) &= -\frac{\pi}{4}a\omega^2 + \frac{\pi}{4}\alpha_1 a + \frac{\pi}{64}\alpha_3(12a^3 + 12a^2b + 12b^2c + 24ab^2 + 24abc + 24ac^2) = 0 \\ \frac{\partial}{\partial b}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) &= -\frac{18\pi b}{8}\omega^2 + \frac{\pi}{4}\alpha_1 b + \frac{\pi}{64}\alpha_3(4b^3 + 24bc^2 + 24abc + 12a^2b + 4a^3 + 24a^2b) = 0 \\ \frac{\partial}{\partial c}\Big(\frac{\partial\widetilde{H}}{\partial\frac{1}{\omega}}\Big) &= -\frac{50\pi c}{8}\omega^2 + \frac{\pi}{4}\alpha_1 c + \frac{\pi}{64}\alpha_3(12c^3 + 24b^2c + 12ab^2 + 12a^2b + 24a^2c) = 0\end{aligned} \tag{3.19}$$

Then, after some simplifications, we obtain

$$a = 0.955091A \qquad b = 0.0430519A \qquad c = 0.0018569A \tag{3.20}$$

Finally, the natural frequency of the system equals to

$$\omega_{THA} = \sqrt{\alpha_1 + 0.7178\alpha_3 A^2} \tag{3.21}$$

4. Discussion and numerical results

The presented solution procedures are used to obtain frequency responses. Variations of the natural frequencies are illustrated in Figs. 3 and 4 for Example 1. The frequency responses are tabulated for some special cases. According to Table 1, it was demonstrated that when the order of the proposed method increases, higher agreement and more accurate results are obtained. The time history obtained for the initial condition is illustrated in Fig. 5. It is seen that in the time domain, a very excellent correlation is still preserved.

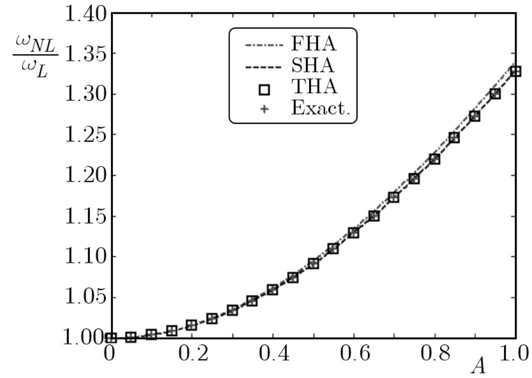

Fig. 3. The frequency ratio (nonlinear/linear) with respect to initial amplitudes for Example 1; $K_1 = 500$, $K_3 = 500$, $m = 50$, $p = 150$, $l = 10$

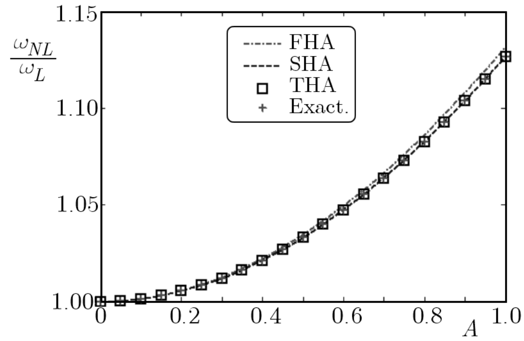

Fig. 4. The frequency ratio (nonlinear/linear) with respect to initial amplitudes for Example 1; $K_1 = 10$, $K_3 = 5$, $m = 1$, $p = 1$, $l = 1$

Table 1. Comparison of approximate and exact frequencies for Example 1.

(m,l,p)	(k_1,k_3)	A	ω_{FHA}	ω_{SHA}	ω_{THA}	ω_{Exact}
(1,1,1)	(10,5)	1	3.2015	3.1877	3.1861	3.1861
(10,10,10)	(10,50)	10	19.3816	18.9985	18.9539	18.9528
(50,25,40)	(30,100)	20	24.5052	24.0202	23.9636	23.9623
(100,50,150)	(70,20)	100	38.7357	37.9687	37.8793	37.8772
(1000,500,1000)	(500,500)	1	0.9332	0.9253	0.92444	0.92442

For Example 2, the numerical results are obtained, and in Table 2 frequency responses of the system are given and analyzed for some special cases. To show and prove the accuracy of these analytical methods, comparisons of analytical and exact results for the practical cases are presented in Fig. 6.

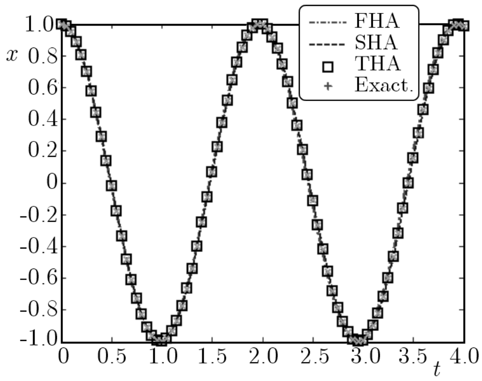

Fig. 5. Time history of dynamic responses $(A = 1,\ M = 1,\ P = 1,\ L = 1,\ K_1 = 10,\ K_3 = 5)$

Table 2. Comparison of approximate and exact frequencies for Example 2

(h,m)	(k_1,k_2)	A	ω_{FHA}	ω_{SHA}	ω_{THA}	ω_{Exact}
(1,1)	(10,5)	1	3.44601	3.4353	3.4340	3.4340
(1,10)	(10,50)	1	1.6955	1.6737	1.6712	1.6711
(10,10)	(30,100)	2	1.7748	1.7731	1.77297	1.77296
(1,100)	(70,20)	5	1.6046	1.5815	1.5789	1.5788

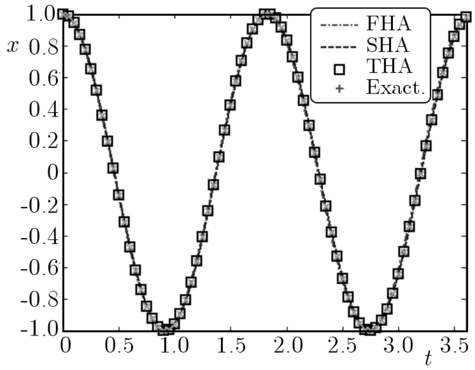

Fig. 6. Time history of dynamic responses $(A = 1,\ m = 1,\ h = 1,\ K_1 = 10,\ K_2 = 5)$

Table 3 reveals the achieved frequency-amplitude relationship for the objective problem of this paper. The results of diverse kinds of approaches are illustrated in this Table using corresponding references.

5. Conclusion

In this paper, two dynamic systems were considered, where in both cases the governing equation was expressed as the Duffing equation. The Hamiltonian approach was then applied in three orders to find the approximate periodic solution of this equation. The accuracy of solution procedures was evaluated by comparing the obtained results with the exact ones in time histories and tables. The effects of nonlinear parameters and initial amplitudes on the natural frequency were also illustrated in two figures. It was proved that as the order of the proposed approach increases, higher agreement and more accurate results are obtained. Indeed, it can be concluded that the higher order Hamiltonian approach is a valid and strong method in evaluating conservative nonlinear oscillatory systems even for large amplitudes and strong nonlinearity. Furthermore, according to Ganji *et al.* (2011), the max-min approach and the Hamiltonian ap-

Table 3. Obtained frequency-amplitude relationship from related references

Approach	Frequency-amplitude relationship
Energy balance method (Younesian *et al.*, 2010a), max-min (Ganji *et al.*, 2011); approach, frequency-amplitude formulation (Younesian *et al.*, 2010a); homotopy perturbation (Younesian *et al.*, 2011); harmonic balance method (Brléndez *et al.*, 2011)	Example 1: $\omega=\sqrt{\Big(k_1-\dfrac{2p}{l}\Big)+\dfrac{3}{4}\Big(k_3-\dfrac{p}{l^3}\Big)A^2}$ Example 2: $\omega=\sqrt{\dfrac{k_1}{m}+\dfrac{3}{8}\dfrac{k_2}{mh^2}A^2}$
Modified energy balance method (Younesian *et al.*, 2011)	Example 1: $\omega=\sqrt{\Big(k_1-\dfrac{2p}{l}\Big)+\dfrac{7}{10}\Big(k_3-\dfrac{p}{l^3}\Big)A^2}$ Example 2: $\omega=\sqrt{\dfrac{k_1}{m}+\dfrac{7}{20}\dfrac{k_2}{mh^2}A^2}$
Simple approach (Ren and He, 2009)	Example 1: $\omega=\sqrt{\Big(k_1-\dfrac{2p}{l}\Big)+\dfrac{7}{9}\Big(k_3-\dfrac{p}{l^3}\Big)A^2}$ Example 2: $\omega=\sqrt{\dfrac{k_1}{m}+\dfrac{7}{18}\dfrac{k_2}{mh^2}A^2}$

proach have the same results for this problem in the first approximation. In addition, basing on Yildirim *et al.* (2012), we can state that the variational approach can lead to similar results for this problem even for a higher order of the approximation. Besides, the harmonic balance method gives the same result for the objective systems (Beléndez *et al.*, 2011). Moreover, results of lots of other papers were developed for the systems considered in this paper, and they are listed in Table 3.

References

1. Askari H., Kalami Yazdi M., Saadatnia Z., 2010, Frequency analysis of nonlinear oscillators with rational restoring force via He's energy balance method and He's variational approach, *Nonlinear Science Letters A*, **1**, 425-430

2. Beléndez A., Arribas E., Francés J., Pascual I., 2011, Notes on application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms, *Mathematical and Computer Modelling*, http://dx.doi.org/10.1016/j.mcm.2011.06.024

3. Cai X.C., Wu W.Y., 2009, He's frequency formulation for the relativistic harmonic oscillator, *Computer and Mathematic with Application*, **58**, 2358-2359

4. Cveticanin L., Kalami Yazdi M., Saadatnia Z., Askari H., 2010, Application of Hamiltonian approach to the generalized nonlinear oscillator with fractional power, *International Journal of Nonlinear Sciences and Numerical Simulation*, **11**, 997-1002

5. Durmaz S., Altay D., Kaya M.O., 2010, High order Hamiltonian approach to nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **11**, 565-570

6. Ganji D.D., Esmaeilpour M., Soleimani S., 2010, Approximate solutions to Van der Pol damped nonlinear oscillators by means of He's energy balance method, *International Journal of Computer Mathematics*, **87** 2014- 2023

7. Ganji D.D., Ranjbar Malidarreh N., Akbarzade M., 2009, Comparison of energy balance period with exact period for arising nonlinear oscillator equations (He's energy balance period

for nonlinear oscillators with and without discontinuities), *Acta Applicandae Mathematicae*, **108**, 353-362

8. Ganji S.S., Barari A., Ganji D.D., 2011, Approximate analysis of two mass-spring systems and buckling of a column, *Computers and Mathematics with Applications*, **61**, 1088-1095
9. Ganji S.S., Ganji D.D., Davodi A.G., Karimpour S., 2010, Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max-min approach, *Applied Mathematical Modelling*, **34**, 2676-2684
10. Ganji S.S., Ganji D.D., Ganji Z.Z., Karimpour S., 2009, Periodic solution for strongly nonlinear vibration systems by He's energy balance method, *Acta Applicandae Mathematicae*, **106**, 79-92
11. He J.H., 2002, Preliminary report on the energy balance for nonlinear oscillations, *Mechanics Research Communications*, **29**, 107-111
12. He J.H., 2006, Some asymptotic methods for strongly nonlinear equations, *International Journal of Modern Physics B*, **20**, 10, 1141-1199
13. He J.H., 2007, Variational approach for nonlinear oscillators, *Chaos Solitons and Fractals*, **34**, 1430-1439
14. He J.H., 2008a, An improved amplitude-frequency formulation for nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **9**, 2, 211-212
15. He J.H., 2008b, Max-min approach to nonlinear oscillators, *International Journal of Nonlinear Sciences and Numerical Simulation*, **9**, 2, 207-210
16. He J.H., 2010, Hamiltonian approach to nonlinear oscillators, *Physics Letters A*, **374**, 2312- 2314
17. Khan Y., Wu Q., Askari H., Saadatnia Z., Kalami Yazdi M., 2010, Nonlinear vibration analysis of a rigid rod on a circular surface via Hamiltonian approach, *Mathematical and Computational Applications*, **15**, 974-977
18. Mehdipour I., Ganji D.D., Mozaffari M., 2010, Application of the energy balance method to nonlinear vibrating equations, **Current Applied Physics**, **10**, 104-112
19. Momeni M., Jamshidi N., Barari A., Ganji D.D., 2011, Application of He's energy balance method to Duffing-harmonic oscillators, *International Journal of Computer Mathematics*, **88**, 1, 135-144
20. Nayfeh A.H., Mook D.T., 1979, *Nonlinear Oscillations*, John Wiley & Sons, New York
21. Ozis T., Yildirim A., Determination of the frequency-amplitude relation for a Duffing-harmonic oscillator by the energy balance method, *Computers and Mathematics with Applications*, **54**, 1184-1187
22. Ren Z.F., He J.H., 2009, A simple approach to nonlinear oscillators, *Physics Letters A*, **373**, 3749-3752
23. Ren Z.-F., Liu G.-Q., Kang Y.-X., Fan H.-Y., Li H.-M., Ren X.-D., Gui W.-K., 2009, Application of He's amplitude-frequency formulation to nonlinear oscillators with discontinuities, *Physica Scripta*, **80**, 4, 045003
24. Rao S.S., 2006, *Mechanical Vibrations Book*, Fourth ed., ISBN: 978-964-9585-5-0
25. Shen Y.Y., Mo L.F., 2009, The max-min approach to a relativistic equation, *Computers and Mathematics with Applications*, **58**, 2131-2133
26. Yazdi M.K., Khan Y., Madani M., Askari H., Saadatnia Z., Yildirim A., Analytical solutions for autonomous conservative nonlinear oscillator, *International Journal of Nonlinear Sciences and Numerical Simulation*, **11**, 11, 979-984
27. Yildirim A., Saadatnia Z., Askari H., 2011a, Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms, *Mathematical and Computer Modelling*, **54**, 697-703

28. Yildirim A., Askari H., Saadatnia Z., Kalami Yazdi M., Khan Y., 2011b, Analysis of nonlinear oscillations of a punctual charge in the electric field of a charged ring via a Hamiltonian approach and the energy balance method, *Computers and Mathematics with Applications*, **62**, 486-490

29. Yildirim A., Saadatnia Z., Askari H., Khan Y., Kalami Yazdi M., 2011c, Higher order approximate periodic solutions for nonlinear oscillators with the Hamiltonian approach, *Applied Mathematics Letters*, **24**, 2042-2051

30. Yildirim A., Askari H., Kalami Yazdi M., Khan Y., 2012, A relationship between three analytical approaches to nonlinear problems, *Applied Mathematics Letters*, doi:10.1016/j.aml.2012.02.001

31. Younesian D., Askari H., Saadatnia Z., KalamiYazdi M., 2010a, Frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency-amplitude formulation and He's energy balance method, *Computer and Mathematic with Application*, **59**, 3222-3228

32. Younesian D., Askari H., Saadatnia Z., Kalami Yazdi M., 2011, Free vibration analysis of strongly nonlinear generalized duffing oscillators using He's variational approach and homotopy perturbation method, *Nonlinear Science Letters A*, **2**, 11-16

33. Younesian D., Askari H., Saadatnia Z., Yildirim A., 2010b, Periodic solutions for the generalized nonlinear oscillators containing fraction order elastic force, *International Journal of Nonlinear Sciences and Numerical Simulation*, **11**, 1027-1032

34. Zeng D.Q., 2009, Nonlinear oscillator with discontinuity by the max-min approach, *Chaos, Solitons and Fractals*, **42**, 2885-2889

35. Zhang Y.N., Xu F., Deng L.L., 2009, Exact solution for nonlinear Schrödinger equation by He's frequency formulation, *Computers and Mathematics with Applications*, **58**, 2449-2451

36. Zhao L., 2009, He's frequency-amplitude formulation for nonlinear oscillators with an irrational force, *Computers and Mathematics with Applications*, **58**, 2477-2479

Zastosowanie metody Hamiltona wyższego rzędu w zagadnieniu drgań układów nieliniowych

Streszczenie

W pracy przedstawiono zastosowanie metody Hamiltona wyższego rzędu do wyznaczania przybliżonych rozwiązań analitycznych dla dwóch nieliniowych układów drgających. Szczegółowej analizie poddano charakterystyki amplitudowo-częstościowe modelu ściskanej belki oraz dyskretnego układu sprężysto-inercyjnego. Otrzymano przybliżone rozwiązania pierwszego, drugiego i trzeciego rzędu, a odpowiedzi częstościowe układów porównano z dokładnymi rezultatami symulacji numerycznych. Na ich podstawie oceniono, że metoda Hamiltona jest stosowalna dla układów nieliniowych, a przybliżenia drugiego i trzeciego rzędu stanowią rozwiązania analityczne o wysokiej dokładności. Uzyskane w pracy wyniki przekonują, że zaproponowana metoda jest prostym i jednocześnie bardzo skutecznym narzędziem rozwiązywania nieliniowych problemów układów mechanicznych o dużym stopniu złożoności.

Manuscript received March 24, 2012; accepted for print June 14, 2012