JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
52, 2, pp. 533-546, Warsaw 2014

OPTIMAL DESIGN OF BAR STRUCTURES WITH THEIR SUPPORTS IN PROBLEMS OF STABILITY AND FREE VIBRATIONS

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The problem of maximization of the buckling load and the problem of maximization of the natural vibration frequency under a condition imposed on the global cost is discussed. Cross-sectional areas of bar structures and number of elastic supports, their positions and stiffnesses (or the number and positions of rigid supports) are selected as design parameters. The proposed here algorithm of optimization of bar structures with their supports is applied for analysis of some optimization problems. Illustrative examples confirm applicability of the proposed approach.

Keywords: buckling load, natural vibration frequency, optimal layout of supports, topological derivative, finite topology modification

1. Introduction

Optimal design of bar structures with respect to cross-sectional dimensions and the number, stiffness and location of supports is a crucial and very complex problem. In particular, in order to improve the structure properties or to avoid resonance, problems of maximization of the buckling load or maximization of the natural vibration frequency with a condition imposed on the global cost can be analyzed. These problems may appear in numerous engineering tasks concerning, for example, design of civil engineering structures, elements of machines and vehicles, aerospace and aerial structures, etc.

The problems of optimal design of bar structures with their supports, because of their importance, were earlier considered in many papers. In the case of beams and frames in regular states they were analyzed in Bojczuk and Mróz (1998) and Mróz and Bojczuk (2003). However, mostly, simplified problems were considered, where for example assumptions of a fixed number of supports or fixed cross-sectional dimensions were used. The general studies of optimal choice of supports and their locations for a given structure were presented by Mróz and Rozvany (1975), Szeląg and Mróz (1978), Mróz (1980), Mróz and Lekszycki (1982), Garstecki and Mróz (1987), etc. The problems of natural vibration frequency maximization and buckling load maximization with respect to stiffness and location of a fixed number of elastic supports were analyzed in Åakeson and Olhoff (1988) and Olhoff and Åakeson (1991). Optimal choice of the supports system from a finite number of possible localizations was analyzed by Zhu and Zhang (2006). Moreover, problems of optimal material distribution in continuous structures with respect to maximization of a chosen eigenfrequency or a gap between successive eigenfrequencies were studied by Du and Olhoff (2006) and Tsai and Cheng (2013). Next, in the case of simultaneous optimal design of bar structures and their supports, optimality conditions and conditions of modification by introduction of new supports to problems of stability and free vibrations were formulated in Bojczuk (2007).

The results obtained here extend the considerations presented in this last paper, i.e. by Bojczuk (2007). The formulation of the problem of maximization of the buckling load with a condition imposed on the global cost, optimality conditions and expressions for sensitivity of the buckling load are presented in Section 2. Analogous considerations for the problem of the smallest or arbitrary chosen eigenfrequency maximization are shown in Section 3. Section 4 is devoted to formulation of conditions of topology modification by introduction of a new support and presentation of algorithms of optimization of bar structures with their supports. Finally, in Section 5, discussion concerning determination of the optimal number of supports depending on boundary conditions and values of cost parameters is performed for some illustrative examples.

2. Problem of maximization of the critical buckling load

2.1. Formulation of the problem

Consider a bar structure composed of K rectilinear elements (segments) of lengths l_k and cross-sectional areas $A(x_k)$. Here x_k, $k = 1, 2, \ldots, K$ denotes the axis attached at the beginning of the k-th member and it coincides with the member axis. The structure is subjected to an external loading $\lambda\mathbf{P}$ increasing proportionally to the load factor λ, where $\mathbf{P}$ denotes the reference load. When a finite element discretization is used, the critical buckling load factor λ_c corresponds to the smallest value from all n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the problem

$$(\mathbf{K} + \lambda_j \mathbf{H})\mathbf{u}_j = \mathbf{0} \tag{2.1}$$

where, $\mathbf{u}_j$, $j = 1, 2, \ldots, n$ are eigenvectors corresponding to eigenvalues λ_j, $j = 1, 2, \ldots, n$, $\mathbf{K}$ is the stiffness matrix and $\mathbf{H}$ denotes the geometric stiffness matrix. The orthogonality condition for the eigenvectors can be assumed in the form

$$\mathbf{u}_k \cdot \mathbf{H}\mathbf{u}_j = b\delta_{kj} \tag{2.2}$$

where δ_{kj} is the Kronecker delta, $(\cdot)$ denotes the scalar product and b is a negative constant. When $b = -1$, Eq. (2.2) also expresses the normalization condition.

Now, the problem of maximization of the buckling load for a bar structure stabilized by an unknown number of transverse elastic supports can be presented in the form

$$\max_{A(x_k), s_i, k_i} \min_{\lambda_j}(\lambda_1, \lambda_2, \ldots, \lambda_n) \quad \text{subject to} \quad C \leqslant C_0 \quad \text{and} \quad (2.1), (2.2) \tag{2.3}$$

where k_i, s_i are respectively the stiffness and the parameter specifying location of the i-th elastic support and C_0 denotes the upper bound imposed on the global cost C. Using the bound formulation (cf. Du and Olhoff, 2007), problem (2.3) becomes

$$\max_{A(x_k), s_i, k_i} \lambda_c \quad \text{subject to} \quad \lambda_j - \lambda_c \geqslant 0 \quad (j = 1, 2, \ldots, n) \quad \text{and} \quad C \leqslant C_0 \quad \text{and} \quad (2.1), (2.2) \tag{2.4}$$

The global cost is assumed as the sum of the structure material cost and cost of supports, namely

$$C = \sum_{k=1}^{K} \int_0^{l_k} c_m EA \, dx_k + \sum_i C_{Si}(k_i) = \sum_{k=1}^{K} \int_0^{l_k} c_m EA \, dx_k + \sum_i (C_{si} + c_{si} k_i) \tag{2.5}$$

where E denotes the Young modulus and c_m is the unit cost of the material. The cost of the i-th support C_{Si} is treated as the sum of two component costs, namely the constant cost of support installation C_{si} and the cost of material proportional to the stiffness k_i, where c_{si} denotes the unit cost of the support material. This form of the support cost function enables analysis of different models, namely: $C_{si} = 0$, $c_{si} > 0$ – the elastic support without installation cost; $C_{si} > 0$, $c_{si} > 0$ – elastic support with installation cost; $C_{si} > 0$, $c_{si} = 0$ – rigid support.

2.2. Optimality conditions

Let us consider problem (2.4) for $I-1$ supports. We introduce Lagrangian in the form

$$L = \lambda_c + \sum_{j=1}^{n} \xi_j(\lambda_j - \lambda_c) + \mu(C_0 - C) \tag{2.6}$$

where $\xi_j(\xi_j \geqslant 0)$, $j = 1,2,\dots,n$, $\mu(\mu \geqslant 0)$ are the Lagrange multipliers. Now, in the case of a structure composed of K segments of constant cross-sectional areas $A_k = \text{const}$, $k = 1,2,\dots,K$, the optimality conditions can be formulated as follows

$$\begin{aligned}
&\frac{\partial L}{\partial \lambda_c} = 1 - \sum_{j=1}^{n} \xi_j = 0 \\
&\frac{\partial L}{\partial A_k} = \sum_{j=1}^{n} \xi_j \frac{\partial \lambda_j}{\partial A_k} - \mu \frac{\partial C}{\partial A_k} = 0 \qquad k = 1,2,\dots,K \\
&\frac{\partial L}{\partial s_i} = \sum_{j=1}^{n} \xi_j \frac{\partial \lambda_j}{\partial s_i} - \mu \frac{\partial C}{\partial s_i} = 0 \qquad i = 1,2,\dots,I-1 \\
&\frac{\partial L}{\partial k_i} = \sum_{j=1}^{n} \xi_j \frac{\partial \lambda_j}{\partial k_i} - \mu \frac{\partial C}{\partial k_i} = 0 \qquad i = 1,2,\dots,I-1 \\
&\xi_j(\lambda_j - \lambda_c) = 0 \qquad j = 1,2,\dots,n \qquad \mu(C_0 - C) = 0
\end{aligned} \tag{2.7}$$

In the case of a variable cross-sectional area $A(x_k)$, using a variational approach, equations $(2.7)_2$ should be substituted by local optimality conditions, where the sensitivities of eigenvalues λ_j can be determined analogously as the sensitivity of the critical value of the load parameter λ_c, see Bojczuk (1999).

If $m(m > 1)$ from the conditions $\lambda_j - \lambda_c \geqslant 0$, $j = 1,2,\dots,n$ are active, we have a multimodal case. Then, the Lagrange multipliers ξ_j connected with the non-active conditions are equal to zero.

Next, in the case, when only one of the conditions $\lambda_j - \lambda_c \geqslant 0$, $j = 1,2,\dots,n$ is active, for example $\lambda_1 - \lambda_c = 0$, we have a unimodal case. Then $\xi_1 = 1$ and the remaining Lagrange multipliers ξ_j are equal to zero, so optimality conditions (2.7) become

$$\begin{aligned}
&\frac{\partial L}{\partial A_k} = \frac{\partial \lambda_1}{\partial A_k} - \mu \frac{\partial C}{\partial A_k} = 0 \qquad k = 1,2,\dots,K \\
&\frac{\partial L}{\partial s_i} = \frac{\partial \lambda_1}{\partial s_i} - \mu \frac{\partial C}{\partial s_i} = 0 \qquad i = 1,2,\dots,I-1 \\
&\frac{\partial L}{\partial k_i} = \frac{\partial \lambda_1}{\partial k_i} - \mu \frac{\partial C}{\partial k_i} = 0 \qquad i = 1,2,\dots,I-1 \\
&\mu(C_0 - C) = 0
\end{aligned} \tag{2.8}$$

Optimality conditions (2.7), (2.8) contain sensitivities of the eigenvalues and global cost with respect to design parameters. They can be determined both, in a continuous form and in the finite element discretization (cf. Bojczuk, 1999, 2001, 2007). In this paper, only the finite element formulation is used, and here the unimodal and multimodal cases of the sensitivity analysis are considered separately.

When the sensitivity expressions are known, the optimal values of design parameters and Lagrange multipliers can be determined from the optimality conditions in the incremental process of gradient optimization.

2.3. Sensitivity of the eigenvalue in the unimodal case

The considerations presented in this Subsection have been prepared using the results obtained by Bojczuk (1999, 2001, 2007). Let us consider the sensitivity of an arbitrary unimodal eigenvalue λ_j with respect to the parameter p, which may correspond to an arbitrary design parameter used in Subsection 2.2, namely A_k, $k = 1, 2, \ldots, K$, and k_i, s_i, $i = 1, 2, \ldots, I-1$. Now, the first derivative of eigenvalue problem (2.1) with respect to the parameter p is

$$\Big(\frac{\partial \mathbf{K}}{\partial p} + \frac{\partial \lambda_j}{\partial p}\mathbf{H} + \lambda_j \frac{\partial \mathbf{H}}{\partial p}\Big)\mathbf{u}_j + (\mathbf{K} + \lambda_j \mathbf{H})\frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{0} \tag{2.9}$$

Next, multiplying (2.9) by the eigenvector $\mathbf{u}_j$, taking into account condition (2.2) and using the finite element discretization, the sensitivity of the unimodal eigenvalue can be presented as follows

$$\frac{\partial \lambda_j}{\partial p} = -\frac{1}{b}\Big[\sum_e \mathbf{u}_{je} \cdot \Big(\frac{\partial \mathbf{K}_e}{\partial p} + \lambda_j N_e \frac{\partial \mathbf{H}_e}{\partial p}\Big)\mathbf{u}_{je} + \lambda_j \sum_e (\mathbf{u}_{je} \cdot \mathbf{H}_e \mathbf{u}_{je})\frac{\partial N_e}{\partial p}\Big] \tag{2.10}$$

where $\mathbf{K}_e$ denotes the stiffness matrix of the e-th element expressed in a global reference system and $\mathbf{u}_{je}$ is the eigenvector for the e-th element. Moreover, $\mathbf{H}_e$ denotes the geometrical stiffness matrix of the e-th element for the unit normal force, expressed in the global reference system, while N_e is the normal force in this element. Here, the phenomenon of normal forces redistribution due to design variation is taken into account. In order to determine the derivatives $\partial N_e/\partial p$ induced by this effect, the sensitivity problem for a linear pre-buckling state should be solved. The details of this approach were presented in Bojczuk (2001, 2007).

2.4. Sensitivity of the eigenvalue in the multimodal case

The considerations presented in this Subsection have been prepared using the results obtained by Bojczuk (1999, 2001, 2007). Let us assume that the $m(m \leqslant n)$ repeated eigenvalues occur, namely

$$\lambda_1 = \lambda_2 = \ldots = \lambda_m \tag{2.11}$$

It is not possible in this case to uniquely distinguish the corresponding eigenvectors. However, we assume that when a small variation in the design parameter p is introduced, the eigenvectors become unique.

Let us choose m arbitrary eigenvectors $\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \ldots, \widehat{\mathbf{u}}_m$ corresponding to the multiple eigenvalue and satisfying condition (2.2). Now, we assume (Mills-Curran, 1987) that the unique eigenvectors can be expressed in the form

$$\mathbf{u}_j = \mathbf{\Phi}\mathbf{a}_j \qquad j = 1, 2, \ldots, m \tag{2.12}$$

where the matrix $\mathbf{\Phi}$ is composed of m columns, namely $\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \ldots, \widehat{\mathbf{u}}_m$, and $\mathbf{a}_j$ denotes the vector describing the transformation from arbitrary chosen eigenvectors $\widehat{\mathbf{u}}_j$, $j = 1, 2, \ldots, m$ to the real eigenvectors $\mathbf{u}_j$, $j = 1, 2, \ldots, m$ appearing after the design variation. So, taking into account that

$$\mathbf{u}_k \cdot \mathbf{H}\mathbf{u}_j = \mathbf{a}_k \cdot (\mathbf{\Phi}^{\mathrm{T}}\mathbf{H}\mathbf{\Phi})\mathbf{a}_j = b\mathbf{a}_k \cdot \mathbf{I}\mathbf{a}_j \tag{2.13}$$

where I denotes the unit matrix, condition (2.2) is satisfied when

$$\mathbf{a}_k \cdot \mathbf{a}_j = \delta_{kj} \tag{2.14}$$

The first derivative of eigenvalue problem (2.1) along the critical state path is expressed by (2.9). So, substituting (2.12) into (2.9) and multiplying by $\mathbf{\Phi}$, we get

$$\Big[\frac{1}{b}\mathbf{\Phi}^{\mathrm{T}}\Big(\frac{\partial\mathbf{K}}{\partial p}+\lambda_j\frac{\partial\mathbf{H}}{\partial p}\Big)\mathbf{\Phi}+\frac{\partial\lambda_j}{\partial p}\mathbf{I}\Big]\mathbf{a}_j=\mathbf{0} \qquad j=1,2,\ldots,m \tag{2.15}$$

Equation (2.15) presents the eigenproblem of m-th order with respect to the eigenvalues $\partial\lambda_j/\partial p$ and eigenvectors $\mathbf{a}_j$. Let us notice that after determination of $\mathbf{a}_j$, the unique eigenvectors can be calculated from (2.12). Moreover, the occurring in (2.15) derivatives of the stiffness matrix $\partial\mathbf{K}/\partial p$ and geometric stiffness matrix $\partial\mathbf{H}/\partial p$ can be for example, determined on the finite element level as it was described by Bojczuk (2001, 2007).

2.5. Topological derivatives with respect to introduction of a new support

Now, we assume, that a new I-th elastic support of the stiffness k_I is introduced at the point x_0 of displacement $w_{j0}=w_j(x_0)$, where $w_j(x)$ denotes the eigenfunction corresponding to an arbitrary unimodal eigenvalue λ_j. Taking into account that the introduction of transverse supports does not influence the geometric stiffness matrix $\mathbf{H}$, the topological derivative of this eigenvalue, in view of (2.10), takes the form

$$\frac{\partial\lambda_j}{\partial k_I}\Big|_{k_I=0}=-\frac{1}{b}\mathbf{u}_{jk}\cdot\frac{\partial\mathbf{K}_k}{\partial k_I}\Big|_{k_I=0}\mathbf{u}_{jk}=-\frac{1}{b}w_{j0}^2 \tag{2.16}$$

where $\mathbf{K}_k$, $\mathbf{u}_{jk}$ are, respectively, the stiffness matrix and displacement vector of the new elastic support for the eigenmode corresponding to the eigenvalue λ_j, while the constant b is defined in (2.2). In the multimodal case, topological derivatives of the multiple eigenvalue can be obtained assuming that the parameter p corresponds to $k_I=0$ and by solving eigenproblem (2.15).

Moreover, taking into account (2.5), the topological derivative of the cost is

$$\frac{\partial C}{\partial k_I}\Big|_{k_I=0}=c_{sI} \tag{2.17}$$

3. Problem of maximization of the natural vibration frequency

3.1. Formulation of the problem

The problem of determination of the natural transverse vibration frequencies ω_j using finite element discretization can be expressed as an appropriate eigenvalue problem with respect to the eigenvalues ω_j^2, $j=1,2,\ldots,n$ and the corresponding eigenvectors $\mathbf{u}_j$, namely

$$(\mathbf{K}+\mathbf{H}-\omega_j^2\mathbf{M})\mathbf{u}_j=\mathbf{0} \tag{3.1}$$

where, as previously, $\mathbf{K}$ is the stiffness matrix, $\mathbf{H}$ geometric stiffness matrix for a fixed load level $\lambda\mathbf{P}$ and $\mathbf{M}$ denotes the mass matrix. The orthogonality and normalization condition for the eigenvectors can be assumed in the form

$$\mathbf{u}_k\cdot\mathbf{M}\mathbf{u}_j=d\delta_{kj} \tag{3.2}$$

where d is a positive constant, and for the normalization condition $d=1$ should be chosen.

Let us now consider the problem of maximization of the first or arbitrary chosen natural transverse vibration frequency ω_j for an initially loaded bar structure stabilized by an unknown number of transverse elastic supports in the form

$$\max_{A(x_k),s_i,k_i}\omega_j \quad \text{subject to} \quad C\leqslant C_0 \quad \text{and} \quad (3.1),(3.2) \tag{3.3}$$

where the notation is the same as in Section 2. When the problem of maximization of the smallest natural vibration frequency ω from all n eigenfrequencies $\omega_1, \omega_2, \dots, \omega_n$ is considered, optimization problem (3.3) can be reformulated as follows

$$\max_{A(x_k), s_i, k_i} \min_{\omega_j}(\omega_1, \omega_2, \dots, \omega_n) \quad \text{subject to} \quad C \leqslant C_0 \quad \text{and} \quad (3.1), (3.2) \tag{3.4}$$

Using the bound formulation (Du and Olhoff, 2007), problem (3.4) becomes

$$\max_{A(x_k), s_i, k_i} \omega \quad \text{subject to} \quad \omega_j - \omega \geqslant 0 \quad (j = 1, 2, \dots, n) \quad \text{and} \quad C \leqslant C_0 \quad \text{and} \quad (3.1), (3.2) \tag{3.5}$$

3.2. Optimality conditions

Let us consider problem (3.5) for $I-1$ supports. We introduce Lagrangian in the form

$$L = \omega + \sum_{j=1}^{n} \xi_j(\omega_j - \omega) + \mu(C_0 - C) \tag{3.6}$$

where $\xi_j(\xi_j \geqslant 0)$, $j = 1, 2, \dots, n$, $\mu(\mu \geqslant 0)$ are the Lagrange multipliers. Now, in the case of a structure composed of K segments of constant cross-sectional areas $A_k = \text{const}$, $k = 1, 2, \dots, K$, the optimality conditions become

$$\begin{aligned}
&\frac{\partial L}{\partial \omega} = 1 - \sum_{j=1}^{n} \xi_j = 0 \\
&\frac{\partial L}{\partial A_k} = \sum_{j=1}^{n} \xi_j \frac{\partial \omega_j}{\partial A_k} - \mu \frac{\partial C}{\partial A_k} = 0 \qquad k = 1, 2, \dots, K \\
&\frac{\partial L}{\partial s_i} = \sum_{j=1}^{n} \xi_j \frac{\partial \omega_j}{\partial s_i} - \mu \frac{\partial C}{\partial s_i} = 0 \qquad i = 1, 2, \dots, I-1 \\
&\frac{\partial L}{\partial k_i} = \sum_{j=1}^{n} \xi_j \frac{\partial \omega_j}{\partial k_i} - \mu \frac{\partial C}{\partial k_i} = 0 \qquad i = 1, 2, \dots, I-1 \\
&\xi_j(\omega_j - \omega) = 0 \qquad j = 1, 2, \dots, n \qquad \mu(C_0 - C) = 0
\end{aligned} \tag{3.7}$$

If $m(m > 1)$ from the conditions $\omega_j - \omega \geqslant 0$, $j = 1, 2, \dots, n$ are active, we have a multimodal case. Then, the Lagrange multipliers ξ_j connected with the non-active conditions are equal zero.

Next, in the case, when only one of the conditions $\omega_j - \omega \geqslant 0$, $j = 1, 2, \dots, n$ is active, for example $\omega_1 - \omega = 0$, we have a unimodal case. Then $\xi_1 = 1$ and the remaining Lagrange multipliers ξ_j are equal zero, so optimality conditions (3.7) take the form

$$\begin{aligned}
&\frac{\partial L}{\partial A_k} = \frac{\partial \omega_1}{\partial A_k} - \mu \frac{\partial C}{\partial A_k} = 0 \qquad k = 1, 2, \dots, K \\
&\frac{\partial L}{\partial s_i} = \frac{\partial \omega_1}{\partial s_i} - \mu \frac{\partial C}{\partial s_i} = 0 \qquad i = 1, 2, \dots, I-1 \\
&\frac{\partial L}{\partial k_i} = \frac{\partial \omega_1}{\partial k_i} - \mu \frac{\partial C}{\partial k_i} = 0 \qquad i = 1, 2, \dots, I-1 \\
&\mu(C_0 - C) = 0
\end{aligned} \tag{3.8}$$

Optimality conditions (3.7), (3.8) contain sensitivities of the eigenfrequencies and global cost with respect to design parameters. They can be determined analogously as in Section 2 using the finite element formulation. Here, the unimodal and multimodal cases of the sensitivity analysis will be considered separately.

3.3. Sensitivity of the eigenfrequency in the unimodal case

The considerations presented in this Subsection have been prepared using the results obtained by Bojczuk (1999, 2001). Let us consider the sensitivity of an arbitrary unimodal eigenfrequency ω_j with respect to the parameter p, which may correspond to an arbitrary design parameter used here, namely A_k, $k = 1, 2, \dots, K$, and k_i, s_i, $i = 1, 2, \dots, I-1$. Now, the first derivative of eigenvalue problem (3.1) with respect to the parameter p can be written as follows

$$\Big(\frac{\partial \mathbf{K}}{\partial p} + \frac{\partial \mathbf{H}}{\partial p} - 2\omega_j \frac{\partial \omega_j}{\partial p}\mathbf{M} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p}\Big)\mathbf{u}_j + (\mathbf{K} + \mathbf{H} - \omega_j^2 \mathbf{M})\frac{\partial \mathbf{u}_j}{\partial p} = \mathbf{0} \tag{3.9}$$

Next, multiplying (3.9) by the eigenvector $\mathbf{u}_j$, taking into account condition (3.2) and using the finite element discretization, the sensitivity of the unimodal eigenfrequency can be presented as follows

$$\frac{\partial \omega_j}{\partial p} = \frac{1}{2\omega_j d}\Big[\sum_e \mathbf{u}_{je} \cdot \Big(\frac{\partial \mathbf{K}_e}{\partial p} + N_e \frac{\partial \mathbf{H}_e}{\partial p} + \frac{\partial N_e}{\partial p}\mathbf{H}_e - \omega_j^2 \frac{\partial \mathbf{M}_e}{\partial p}\Big)\mathbf{u}_{je}\Big] \tag{3.10}$$

where $\mathbf{K}_e$, $\mathbf{H}_e$, $\mathbf{u}_{je}$, N_e are described in Subsection 2.3, and $\mathbf{M}_e$ denotes the mass matrix of the e-th element expressed in the global reference system.

3.4. Sensitivity of the eigenfrequency in the multimodal case

The considerations presented in this Subsection have been prepared using results obtained by Bojczuk (1999, 2001). Let us assume that the $m(m \leqslant n)$ repeated eigenfrequencies occur, namely

$$\omega_1 = \omega_2 = \ldots = \omega_m \tag{3.11}$$

It is not possible in this case to uniquely distinguish the corresponding eigenvectors. However, we assume that when a small variation of the design parameter p is introduced, the eigenvectors become unique.

Using the approach described by Eq. (2.12)-(2.14) and analogous notation as in Subsection 2.4, but substituting (2.13) by

$$\mathbf{u}_k \cdot \mathbf{M}\mathbf{u}_j = \mathbf{a}_k \cdot (\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi})\mathbf{a}_j = d\mathbf{a}_k \cdot \mathbf{I}\mathbf{a}_j \tag{3.12}$$

finally, we get

$$\Big[\frac{1}{2\omega_j d}\mathbf{\Phi}^{\mathrm{T}}\Big(\frac{\partial \mathbf{K}}{\partial p} + \frac{\partial \mathbf{H}}{\partial p} - \omega_j^2 \frac{\partial \mathbf{M}}{\partial p}\Big)\mathbf{\Phi} - \frac{\partial \omega_j}{\partial p}\mathbf{I}\Big]\mathbf{a}_j = \mathbf{0} \qquad j = 1, 2, \dots, m \tag{3.13}$$

Equation (3.13) presents an eigenproblem of the m-th order with respect to the eigenvalues $\partial\omega_j/\partial p$ and eigenvectors $\mathbf{a}_j$. Let us notice that after determination of $\mathbf{a}_j$, the unique eigenvectors can be calculated from (2.12).

3.5. Topological derivatives of eigenfrequency with respect to introduction of a new support

Now, we assume, that a new I-th elastic support of the stiffness k_I is introduced at the point x_0 of displacement $w_{j0} = w_j(x_0)$, where $w_j(x)$ denotes the eigenfunction corresponding to an arbitrary unimodal eigenfrequency ω_j. Taking into account that the introduction of transverse supports does not influence the geometric stiffness matrix $\mathbf{H}$ and the mass matrix $\mathbf{M}$, the topological derivative of this eigenfrequency, in view of (3.10), takes the form

$$\frac{\partial \omega_j}{\partial k_I}\Big|_{k_I=0} = \frac{1}{2\omega_j d}\mathbf{u}_{jk} \cdot \frac{\partial \mathbf{K}_k}{\partial k_I}\Big|_{k_I=0}\mathbf{u}_{jk} = \frac{1}{2\omega_j d}w_{j0}^2 \tag{3.14}$$

where $\mathbf{K}_k$, $\mathbf{u}_{jk}$ are, respectively, the stiffness matrix and displacement vector of the new elastic support for the eigenmode corresponding to the eigenfrequency ω_j, while the constant d is defined in (3.2). In the multimodal case, analogously as in Subsection 2.5, topological derivatives of the multiple eigenfrequency can be obtained assuming that the parameter p corresponds to $k_I = 0$ and by solving eigenproblem (3.13).

4. Topology modification conditions and the algorithm of optimization

4.1. Topology modification conditions by the introduction of a new support

After determination of the optimal values of design parameters for fixed topology, next we try to introduce a new support or supports (Mróz and Bojczuk, 2003).

At first, for the problem of buckling load maximization, we consider the introduction of the I-th transverse elastic support of zero stiffness $k_I = 0$. Then, using the topological derivative, the condition of introduction of this modification for the unimodal case, assuming that the critical load factor is equal to λ_1, can be written analogously to $(2.7)_4$, namely

$$\frac{\partial L}{\partial k_I}\bigg|_{k_I=0} = \frac{\partial \lambda_1}{\partial k_I}\bigg|_{k_I=0} - \mu \frac{\partial C}{\partial k_I}\bigg|_{k_I=0} > 0 \tag{4.1}$$

so it corresponds to the positive value of the topological derivative of the Lagrangian. Taking into account (2.16) for $j = 1$ and (2.17), this condition can be presented in the form

$$|w_{10}| > \sqrt{\mu c_{sI} |b|} \tag{4.2}$$

It is important to notice that when condition (4.2) is satisfied, the new support should be introduced at the point x_0 of the maximal value of displacement $|w_{10}| = |w_1(x_0)|$, where $w_1(x)$ denotes the eigenfunction corresponding to the critical load factor λ_1.

Next, let us consider the multimodal case with $m(m \leqslant n)$ repeated eigenvalues. Then, the condition of the I-th support introduction of zero stiffness, analogously to $(2.7)_4$, can be written as

$$\frac{\partial L}{\partial k_I}\bigg|_{k_I=0} = \sum_{j=1}^{m} \xi_j \frac{\partial \lambda_j}{\partial k_I}\bigg|_{k_I=0} - \mu \frac{\partial C}{\partial k_I}\bigg|_{k_I=0} > 0 \tag{4.3}$$

where the corresponding topological derivatives can be determined as it was described in Subsection 2.5.

Moreover, let us consider the case of introduction of a support of a finite stiffness $k_I(k_I > 0)$. The condition of this modification takes the form

$$\lambda_c^{(m)} - \lambda_c^{(p)} > 0 \tag{4.4}$$

where $\lambda_c^{(p)}$ denotes the value of the critical load before modification, and $\lambda_c^{(m)}$ is the value of this load after modification. It is assumed that after modification, all supports are located only in nodal points of successive eigenmodes. Next, along the path of the constant cost $C = C_0$, we determine the optimal values of design parameters A_k, $k = 1, 2, \ldots, K$, and k_i, s_i, $i = 1, 2, \ldots, I - 1$ in the process of standard gradient optimization. Let us notice that this approach can be used not only for elastic supports but also for rigid supports.

Finally, let us consider the problem of maximization of the smallest natural transverse vibration frequency ω_1. In this case, the corresponding modification conditions can be written analogously to (4.1)-(4.4). So, the condition of the I-th support introduction of zero stiffness $k_I = 0$ in the unimodal case is

$$\frac{\partial L}{\partial k_I}\Big|_{k_I=0} = \frac{\partial \omega_1}{\partial k_I}\Big|_{k_I=0} - \mu \frac{\partial C}{\partial k_I}\Big|_{k_I=0} > 0 \qquad \text{or} \qquad |w_{10}| > \sqrt{2\mu\omega_1 c_{sI} d} \tag{4.5}$$

and in the multimodal case takes the form

$$\frac{\partial L}{\partial k_I}\Big|_{k_I=0} = \sum_{j=1}^{m} \xi_j \frac{\partial \omega_j}{\partial k_I}\Big|_{k_I=0} - \mu \frac{\partial C}{\partial k_I}\Big|_{k_I=0} > 0 \tag{4.6}$$

Next, the condition of introduction of the support of a finite stiffness or of a rigid support can now be presented as follows

$$\omega_1^{(m)} - \omega_1^{(p)} > 0 \tag{4.7}$$

4.2. Algorithm of optimization

On the basis of the presented considerations, the following algorithm for the optimization of bar structures with their supports can be proposed, namely:

1) Choose the initial design with a required but relatively small number of elastic and/or rigid supports.
2) Using an algorithm of gradient optimization, determine the optimal values of cross-sectional areas A_k, $k = 1, 2, \ldots, K$ and positions and stiffnesses of the supports k_i, s_i, $i = 1, 2, \ldots, I-1$.
3) Check, depending on the problem, if conditions (4.2) or (4.5) ((4.3) or (4.6)) of the infinitesimally small structure modification by the introduction of a new support is satisfied. When the condition is fulfilled, introduce the support and return to 2). Otherwise, go to the next point.
4) Check, depending on the problem, if condition (4.4) or (4.7) of the finite structure modification by the introduction of the new support is satisfied. When the condition is fulfilled, introduce the support and return to 2). Otherwise, go to the next point.
5) If any condition of the modification is not satisfied, the last obtained design treat as the optimal and terminate the procedure.

5. Examples of optimal design of beams with their supports and discussion on the solutions

Numerical examples of optimal design of bar structures with their supports presented in this Section illustrate applicability and usefulness of the proposed approach.

In particular, in the case of rigid supports of a finite stiffness and constant cost, called in this paper the installation cost, only finite modifications can be used for which the conditions of the introduction are expressed by (4.4) and (4.7). The optimal solutions correspond then to the introduction of supports of a finite stiffness in the nodal points of successive eigenmodes.

In examples analyzed in the later part of this Section, the elastic supports with a negligible installation cost, are assumed. Taking into account that these problems are not as trivial as in the case of the supports of a finite stiffness, discussion on the solutions according to the cost parameter is conducted.

5.1. Optimization of a simply supported beam with its additional elastic support with respect to maximization of the buckling load

Consider optimal design problem (2.4) for a simply supported prismatic strut of length l (Fig. 1a). The aim is to determine the circular cross-section area and the number, stiffness and location of elastic supports stabilizing the strut in order to maximize the buckling load with a constraint set on the global cost. Assume that the cost of supports at the ends of the strut is constant and is not taken into account in further considerations.

The data for the strut are as follows: the length $l = 8\,\mathrm{m}$, the initial moment of inertia $I^{(0)} = 10^{-5}\,\mathrm{m}^4$, the initial cross-section area $A^{(0)} = \sqrt{4\pi I^{(0)}} = 1.12071 \cdot 10^{-2}\,\mathrm{m}^2$, the Young modulus $E = 2.1 \cdot 10^5\,\mathrm{MPa}$, the specific material cost $c_m = 1\,(\mathrm{Nm})^{-1}$, the initial cost of the structure $C = c_m E A^{(0)} l = 1.883277 \cdot 10^{10}$ corresponding to the upper bound C_0 imposed on this cost, the installation cost of each elastic support $C_s = 0$ and the reference load $P = 1$ (Fig. 1a).

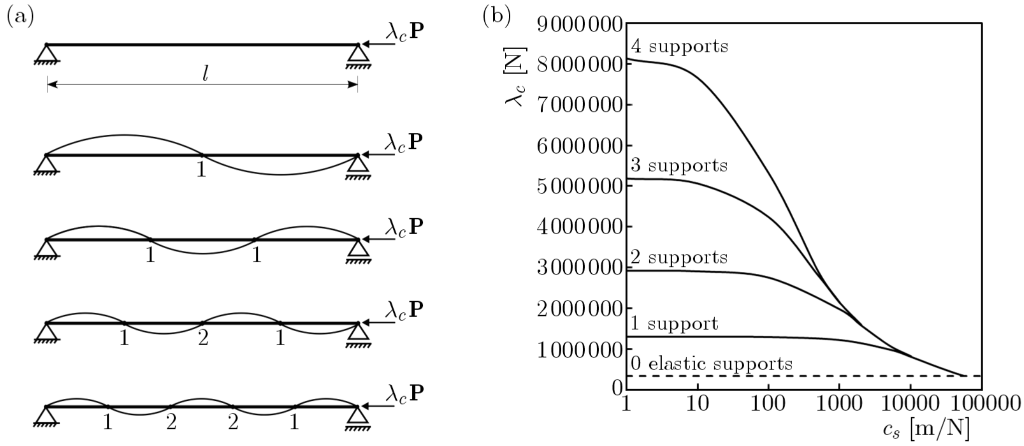

Fig. 1. (a) Buckling of the strut – successive eigenmodes; (b) optimal value of the critical load for different numbers of elastic supports according to the unit cost c_s

Now, we discuss the problem of optimal design for different values of the unit cost c_s, yet the same for all supports. The analysis was limited to four additional elastic supports and carried out according to the algorithm described in Subsection 4.2. In the case $c_s = 0$, the values of the critical load for the number of supports n growing from zero are respectively $\lambda_c^{(0)} = 3.2385 \cdot 10^5$ N, $\lambda_c^{(1)} = 2^2\lambda_c^{(0)}, \ldots, \lambda_c^{(n)} = (n+1)^2\lambda_c^{(0)}$.

Figure 1b shows the relationship between the maximum critical load λ_c and the unit cost of the support c_s for different numbers of elastic supports. Moreover, selected results of the optimization for different numbers of supports and different values of the unit cost are presented in Table 1, where numeration of the supports and their locations are given according to Fig. 1a.

All presented solutions correspond to multimodal states, usually to bimodal states. Let us notice that for any $c_s > 0$, the optimal solution with a finite number of elastic supports or without elastic supports always exists. There is a following relation here that for smaller values of c_s, the optimal design contains a bigger number of additional elastic supports. If for an optimization problem with a certain fixed number of elastic supports there exists the optimal solution with a smaller number of these supports, the solution of this problem is numerically unstable or tends to the mentioned optimal solution with a reduced number of the supports.

For a beam with two additional elastic supports, a comparison of the critical load $\lambda_c^{(z)}$ corresponding to the introduction in the nodal points of eigenmodes the elastic supports of the smallest stiffness, for which the support deflections are equal to zero (Timoshenko and Gere, 1963) with optimal values of the critical load λ_c determined in this paper, is done. The results are presented in Table 2.

Table 1

No. of supports	c_s [m/N]	λ_c [N]	A [m^2]	k_1 [N/m]	l_1 [m]	k_2 [N/m]	l_2 [m]
1	1	$1.30 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$6.48 \cdot 10^5$	4.0		
	10^1	$1.29 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$6.47 \cdot 10^5$	4.0		
	10^2	$1.29 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$6.43 \cdot 10^5$	4.0		
	10^3	$1.21 \cdot 10^6$	$1.08 \cdot 10^{-2}$	$6.07 \cdot 10^5$	4.0		
	$5 \cdot 10^3$	$9.80 \cdot 10^5$	$9.75 \cdot 10^{-3}$	$4.90 \cdot 10^5$	4.0		
	10^4	$8.02 \cdot 10^5$	$8.82 \cdot 10^{-3}$	$4.01 \cdot 10^5$	4.0		
	$2 \cdot 10^4$	$6.01 \cdot 10^5$	$7.63 \cdot 10^{-3}$	$3.00 \cdot 10^5$	4.0		
	$4 \cdot 10^4$	$4.11 \cdot 10^5$	$6.32 \cdot 10^{-3}$	$2.06 \cdot 10^5$	4.0		
	$5.6 \cdot 10^4$	$3.32 \cdot 10^5$	$5.68 \cdot 10^{-3}$	$1.66 \cdot 10^5$	4.0		
	$5.7 \cdot 10^4$	$3.29 \cdot 10^5$	$1.12 \cdot 10^{-3}$	–	–		
2	1	$2.91 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$41.66 \cdot 10^5$	2.67		
	10^1	$2.90 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$32.12 \cdot 10^5$	2.66		
	10^2	$2.74 \cdot 10^6$	$1.09 \cdot 10^{-2}$	$26.63 \cdot 10^5$	2.56		
	10^3	$1.97 \cdot 10^6$	$9.44 \cdot 10^{-3}$	$14.89 \cdot 10^5$	2.35		
	$2 \cdot 10^3$	$1.57 \cdot 10^6$	$8.64 \cdot 10^{-3}$	$10.80 \cdot 10^5$	2.27		
	$5 \cdot 10^3$	$1.08 \cdot 10^6$	$8.46 \cdot 10^{-3}$	$4.61 \cdot 10^5$	2.34		
	10^4	$8.22 \cdot 10^5$	$8.32 \cdot 10^{-3}$	$2.43 \cdot 10^5$	2.91		
3	1	$5.18 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$80.00 \cdot 10^5$	2.00	$97.87 \cdot 10^5$	4.0
	10^1	$5.05 \cdot 10^6$	$1.11 \cdot 10^{-2}$	$73.51 \cdot 10^5$	1.98	$94.29 \cdot 10^5$	4.0
	10^2	$4.23 \cdot 10^6$	$1.02 \cdot 10^{-2}$	$53.65 \cdot 10^5$	1.91	$58.73 \cdot 10^5$	4.0
	$5 \cdot 10^2$	$2.79 \cdot 10^6$	$8.76 \cdot 10^{-3}$	$29.01 \cdot 10^5$	1.84	$24.45 \cdot 10^5$	4.0
	10^3	$2.12 \cdot 10^6$	$8.24 \cdot 10^{-3}$	$18.89 \cdot 10^5$	1.87	$12.06 \cdot 10^5$	4.0
	$2 \cdot 10^3$	$1.58 \cdot 10^6$	$8.23 \cdot 10^{-3}$	$11.27 \cdot 10^5$	2.09	$2.48 \cdot 10^5$	4.0
4	1	$8.08 \cdot 10^6$	$1.12 \cdot 10^{-2}$	$151.40 \cdot 10^5$	1.58	$193.48 \cdot 10^5$	3.21
	10^1	$7.61 \cdot 10^6$	$1.09 \cdot 10^{-2}$	$129.22 \cdot 10^5$	1.55	$171.97 \cdot 10^5$	3.23
	10^2	$5.29 \cdot 10^6$	$9.43 \cdot 10^{-3}$	$74.55 \cdot 10^5$	1.49	$82.43 \cdot 10^5$	3.25
	$5 \cdot 10^2$	$2.89 \cdot 10^6$	$8.08 \cdot 10^{-3}$	$30.33 \cdot 10^5$	1.57	$22.19 \cdot 10^5$	3.35
	10^3	$2.13 \cdot 10^6$	$8.09 \cdot 10^{-3}$	$18.51 \cdot 10^5$	1.78	$7.72 \cdot 10^5$	3.52

Table 2

c_s [m/N]	10	100	1000	2000	5000	10 000	20 000
$\frac{\lambda_c-\lambda_c^{(z)}}{\lambda_c} \cdot 100\%$	0.075	0.584	8.660	14.565	26.080	40.057	53.342

5.2. Optimization of a clamped beam and its additional elastic support with respect to maximization of the smallest natural frequency of transverse vibrations

Consider optimal design problem (3.5) for a unilaterally clamped prismatic bar of length l (Fig. 2a). The aim is to determine the circular cross-section area of bar and the number, stiffness and location of elastic supports in order to maximize the smallest eigenfrequency of transverse vibrations, under a constraint imposed on the global cost. Assume that the cost of the attachment is constant and is not taken into account in further considerations.

The data for the bar are as follows: the length $l = 4\,\mathrm{m}$, the initial moment of inertia $I^{(0)} = 4.90874 \cdot 10^{-5}\,\mathrm{m}^4$, the initial cross-section area $A^{(0)} = \sqrt{4\pi I^{(0)}} = 7.854 \cdot 10^{-3}\,\mathrm{m}^2$, the Young modulus $E = 2.1 \cdot 10^5\,\mathrm{MPa}$, the specific material cost $c_m = 1\,(\mathrm{Nm})^{-1}$, the density of material $\rho = 7800\,\mathrm{kg/m}^3$, the initial cost of the structure $C = c_m E A^{(0)} l = 6.5973 \cdot 10^9$ corresponding to the upper bound C_0 imposed on this cost, the installation cost of each elastic support $C_s = 0$.

Now, we discuss the problem of optimal design for different values of the unit cost c_s. The analysis was limited to four additional elastic supports and carried out according to the algorithm described in Subsection 4.2. In the case $c_s = 0$, the maximum values of vibration eigenfrequencies, for the number of additional elastic supports growing from zero and ensuring zero support deflection of successive eigenmodes, are respectively $\omega_0 = 28.5\,\mathrm{s}^{-1}$, $\omega_1 = 178.7\,\mathrm{s}^{-1}$, $\omega_2 = 500.5\,\mathrm{s}^{-1}$, $\omega_3 = 982.4\,\mathrm{s}^{-1}$, $\omega_4 = 1629.7\,\mathrm{s}^{-1}$, etc.

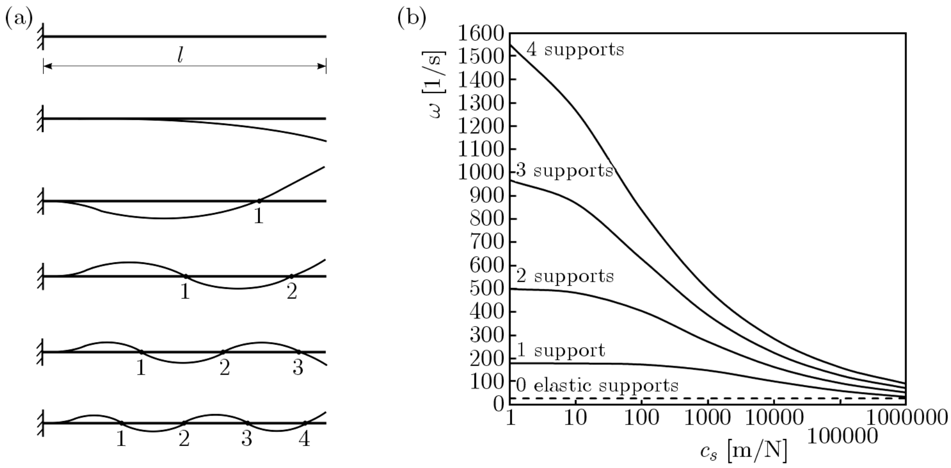

Fig. 2. (a) Free vibrations of the clamped beam – successive eigenmodes; (b) optimal value of the smallest vibration eigenfrequency for different numbers of elastic supports according to the unit cost c_s

Figure 2b shows the relationship between the maximum value of the smallest vibration eigenfrequency ω and the unit cost of the support c_s for different numbers of supports. Moreover, selected results of the optimization for different numbers of the elastic supports and different values of the unit cost are presented in Table 3, where numeration of the supports and their locations are given according to Fig. 2a.

All presented solutions, as in the former example, correspond to multimodal states, usually to bimodal states. The optimally placed elastic supports are located in the nodal points of eigenmodes or in their close neighborhood.

Let us notice that for a relatively high value of c_s we obtain a solution which corresponds to the structure with a bar of a relatively small cross-section area and supports of a relatively high stiffness. This solution has little practical significance. In general, with the growing number of supports for the given c_s the optimal solution gives structures of the growing minimal vibration eigenfrequency. One can expect that the solution in the limit case tends to Winkler's foundation. In order to obtain optimal solutions of practical significance containing a finite number of supports, the condition imposed on the minimum cross-section area can be introduced or the installation cost can be taken into account.

Consider a beam with two additional elastic supports, and let $\omega^{(z)}$ denote the smallest vibration eigenfrequency corresponding to the introduction, at the nodal points of eigenmodes, elastic supports of the smallest stiffness, for which the deflections are equal to zero (Åakeson and Olhoff, 1988). The results of comparison of $\omega^{(z)}$ with the optimal values of the smallest vibration eigenfrequency ω determined in this paper are presented in Table 4.

6. Concluding remarks

The problem of maximization of the buckling load and the problem of maximization of the smallest or arbitrary chosen natural vibration frequency with a condition imposed on the global cost is formulated in this paper. The optimality conditions, sensitivity expressions and topology modification conditions by introduction of new supports are derived. On this basis, a uniform

Table 4

c_s [m/N]	10	100	1000	10 000	100 000	200 000
$\frac{\lambda_c-\lambda_c^{(z)}}{\lambda_c}\cdot 100\%$	0.25	0.59	1.17	1.54	1.67	1.69
c_s [m/N]	400 000	800 000	1 600 000	3 200 000	6 400 000	
$\frac{\lambda_c-\lambda_c^{(z)}}{\lambda_c}\cdot 100\%$	1.70	1.71	1.71	1.72	1.72	

Table 3

No. of supports	c_s [m/N]	ω [1/s]	A [m^2]	k_1 [N/m]	l_1 [m]	k_2 [N/m]	l_2 [m]	k_3 [N/m]	l_3 [m]	k_4 [N/m]	l_4 [m]
1	1	178.6	$7.85\cdot10^{-3}$	$42.93\cdot10^{5}$	3.134						
	10^1	178.1	$7.80\cdot10^{-3}$	$42.43\cdot10^{5}$	3.134						
	10^2	173.4	$7.40\cdot10^{-3}$	$38.15\cdot10^{5}$	3.134						
	10^3	148.4	$5.42\cdot10^{-3}$	$20.46\cdot10^{5}$	3.134						
	10^4	101.5	$2.53\cdot10^{-3}$	$4.47\cdot10^{5}$	3.134						
	10^5	60.97	$9.15\cdot10^{-4}$	$5.83\cdot10^{4}$	3.134						
	10^6	35.02	$3.02\cdot10^{-4}$	$6.34\cdot10^{3}$	3.134						
2	1	498.4	$7.79\cdot10^{-3}$	$320.7\cdot10^{5}$	2.014	$234.6\cdot10^{5}$	3.471				
	10^1	481.9	$7.28\cdot10^{-3}$	$283.6\cdot10^{5}$	2.012	$195.0\cdot10^{5}$	3.474				
	10^2	403.3	$5.14\cdot10^{-3}$	$135.6\cdot10^{5}$	2.007	$92.61\cdot10^{5}$	3.482				
	10^3	271.3	$2.36\cdot10^{-3}$	$27.5\cdot10^{5}$	2.007	$18.63\cdot10^{5}$	3.489				
	10^4	162.3	$8.54\cdot10^{-4}$	$3.51\cdot10^{5}$	2.008	$2.37\cdot10^{5}$	3.492				
	10^5	93.08	$2.82\cdot10^{-4}$	$3.80\cdot10^{4}$	2.009	$2.56\cdot10^{4}$	3.493				
	10^6	52.67	$9.05\cdot10^{-5}$	$3.89\cdot10^{3}$	2.009	$2.63\cdot10^{3}$	3.493				
3	1	966.0	$7.58\cdot10^{-3}$	$807.5\cdot10^{5}$	1.432	$902.3\cdot10^{5}$	2.576	$582.8\cdot10^{5}$	3.624		
	10^1	866.8	$6.12\cdot10^{-3}$	$518.4\cdot10^{5}$	1.431	$566.1\cdot10^{5}$	2.576	$369.4\cdot10^{5}$	3.628		
	10^2	626.2	$3.24\cdot10^{-3}$	$140.3\cdot10^{5}$	1.431	$148.7\cdot10^{5}$	2.577	$98.5\cdot10^{5}$	3.633		
	10^3	385.8	$1.25\cdot10^{-3}$	$20.25\cdot10^{5}$	1.433	$21.13\cdot10^{5}$	2.579	$14.13\cdot10^{5}$	3.636		
	10^4	223.5	$4.21\cdot10^{-4}$	$2.28\cdot10^{5}$	1.434	$2.37\cdot10^{5}$	2.580	$1.59\cdot10^{5}$	3.637		
	10^5	126.9	$1.36\cdot10^{-4}$	$2.38\cdot10^{4}$	1.434	$2.46\cdot10^{4}$	2.580	$1.65\cdot10^{4}$	3.637		
	10^6	71.54	$4.33\cdot10^{-5}$	$2.40\cdot10^{3}$	1.435	$2.48\cdot10^{3}$	2.580	$1.67\cdot10^{3}$	3.637		
4	1	1551.5	$7.15\cdot10^{-3}$	$1524.1\cdot10^{5}$	1.113	$1620.8\cdot10^{5}$	1.999	$1681.7\cdot10^{5}$	2.893	$1092.3\cdot10^{5}$	3.709
	10^1	1266.8	$4.80\cdot10^{-3}$	$669.8\cdot10^{5}$	1.112	$698.0\cdot10^{5}$	2.001	$721.0\cdot10^{5}$	2.893	$473.3\cdot10^{5}$	3.713
	10^2	836.5	$2.13\cdot10^{-3}$	$127.4\cdot10^{5}$	1.114	$130.4\cdot10^{5}$	2.004	$134.1\cdot10^{5}$	2.893	$89.05\cdot10^{5}$	3.716
	10^3	496.6	$7.57\cdot10^{-4}$	$15.87\cdot10^{5}$	1.115	$16.13\cdot10^{5}$	2.006	$16.56\cdot10^{5}$	2.894	$11.06\cdot10^{5}$	3.717
	10^4	284.2	$2.49\cdot10^{-4}$	$1.70\cdot10^{5}$	1.116	$1.73\cdot10^{5}$	2.006	$1.77\cdot10^{5}$	2.894	$1.19\cdot10^{5}$	3.717
	10^5	160.7	$7.97\cdot10^{-5}$	$1.74\cdot10^{4}$	1.116	$1.76\cdot10^{4}$	2.007	$1.81\cdot10^{4}$	2.894	$1.21\cdot10^{4}$	3.717
	10^6	90.51	$2.53\cdot10^{-5}$	$1.76\cdot10^{3}$	1.116	$1.78\cdot10^{3}$	2.007	$1.82\cdot10^{3}$	2.894	$1.22\cdot10^{3}$	3.717

heuristic algorithm for optimal design is presented taking into account cross-sectional areas of bar elements of the considered structure, the number and position of supports and their stiffnesses in the case of elastic supports. Using this algorithm, a detailed analysis of the number and arrangement of additional elastic supports, depending on cost parameters, is performed for some test problems, and it is shown that the optimal solutions correspond to multimodal states, usually to bimodal states.

The presented approach can be applied to problems with other cost functions of the supports. It can also be used to problems with damped or forced vibrations.

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Manuscript received August 5, 2013; accepted for print December 1, 2013